**Response to Reviewer 1 (Dr. Daniel Chavas)**

We would like to thank Reviewer 1 for your constructive comments and suggestions. In this revision, we have made a number of changes to take into account your concerns. Please find below our point-by-point responses to your comments and our corresponding changes.

Overall this work presents a very compelling, albeit mathematically complex, physical argument in a simplified barotropic framework for the upper bound on the number of tropical cyclones that emerge from the breakdown of the ITCZ. This upper bound may have a direct role in setting the annual number of tropical cyclones on Earth, for which no current theory exists. I find the manuscript to be of very high quality in terms of both writing and physical framework, though I cannot fully evaluate the mathematical analysis, particularly for the Principle of Exchange of Stabilities, as it lies beyond the scope of my expertise. Nonetheless, if fully validated, this would appear to be a rather remarkable feat of dynamical systems theory for explaining the nature of the breakdown of the ITCZ on an Earth-like planet and its potential relevance.

Your encouraging evaluation of our manuscript is much appreciated. We hope that our revision below will be now acceptable to you.

**Specific comments:**

P2L30: It would seem very relevant here to include the work of Patricola et al. (2018; https://agupubs.onlinelibrary.wiley.com/doi/pdf/10.1002/2017GL076081), which found that filtering out AEWs did not alter the number of storms in the Atlantic, suggesting that disturbances from other sources (e.g. ITCZ breakdown) may take their place.

Thank you for pointing out to us this work of Patricola et al. (2018), which is indeed relevant to our study. Their finding about the insignificant role of AEWs in modulating the number of the TC climatology in the Atlantic basin is noteworthy, as it is consistent with our model presented in this study in the sense that our model does not contain any local feature such as the African Jet that triggers AEWs. One thing that we wish to note is that the AEWs are just one class of a much broader group of easterly waves in the tropical region. While AEWs can be filtered as presented in Patricola et al. (2018), the other modes of easterly waves such as mixed equatorial Robssy waves can still exist. Anyway, both Petricola et al.'s results and our study herein appear to suggest an intrinsic mechanism at the large scale, which controls the climatology of the TC numbers beyond the basin-specific features. This work has been now cited in this revision.

P3L5/Figure 1: I'm not sure this figure, taken from Kieu et al. (2018), is appropriate here, as it is included without explanation. If the figure is presented within the stated reference then the reference alone should suffice without the figure being reproduced.

Our purpose of including this figure here is to highlight the key importance of episodic development of TCs at the global scale so one can estimate the global number of TCs annually. Essentially, we need two pieces of information to be able to determine how many TCs the tropical atmosphere can support annually, which include 1) the maximum number of TCs that the tropical atmosphere can produce for any given episode of TC formation, and 2) the frequency that the new episode of TC formation will occur. The analytical work herein addresses the first question, and the global modelling of TC formation shown in Figure 1 addresses the second question. Given that we plan to have an upcoming study that specifically tackle the second question in much more details, we have removed Figure 1 in this work per your suggestion.

P11L25: I'm unsure where "3 x 3000 km" is coming from here. Lx/m = 3333 km.

Thank you. This is our typo. We really meant 3000 km, not 3x3000 km here. This has been fixed.

P11L27: I'm not sure I agree with this statement that storm size "must be larger than a limit of ~10ˆ3 km" – this may be a lower bound on size for this specific case of an equatorial band of TCs of equal size. In reality individual storms may take a range of sizes, and certainly there are instances of very small storms that appear to be have a diameter much smaller than this length scale. This is not incompatible with the model presented here, reality is simply more complex. Given that it is an important parameter (aspect ratio, with Lx fixed), how would one plausibly define Ly for the real world? Is there some physical sense of what would represent the poleward boundary relevant to the system?

Our main point in this discussion is that if all high zonal wavenumbers must be stable as found in this study, then any disturbance corresponding to a large wavenumber (i.e., a small size) that could potentially grow into a TC would not occur. So, only those with $m < 12$ (i.e., their diameters are > 3000 km) can have a chance, which agree well with the typical scale of a region where a TC emerges in the tropical region. Of course, this by no means eliminates the existence of a small TC such as midgets at the higher latitudes, because our analytical results can only provide an estimate for the size of the" hot spot" where a TC disturbance can develop. Talking about the size of a fully-developed TC is beyond our current work, as it involves various complex factors as pointed out in your several recent studies on the topic of the TC size. We have revised this discussion to avoid misleading impression.

Regarding the width of the tropical region, it is indeed hard to be precisely determined. We simply use a typical value of $20^0$ for the tropical channel, based on the definition of the tropics up to the tropic of Capricorn (~23.5°). Additional analyses for a few different widths from 15-20º do not show much difference, because the zonal scale is, after all, always an order of magnitude larger than the meridional scale.

P14L34: Aren't these westward-moving disturbances simply Rossby waves? Based on Fig 2, at the ITCZ location (dashed line), Uyy=0 and thus the PV gradient is purely beta. Does their phase speed follow the barotropic Rossby wave phase speed for wavenumber equal to the unstable wavenumber predicted by the model (4 m/s)? I wonder if Rossby waves are a more appropriate analog than African Easterly Waves for the features in the model.

We totally agree. It is entirely possible that that easterly waves here are a mode of the equatorial mixed Rossby waves, because the spatial scale as well as the phase speed are consistent with westward-moving Rossby wave (meridional mode $n$=1). However, we also wish to caution here that the numerical procedure of finding the unstable mode on the central manifold presented in this section does not allow us to separate different modes of easterly waves. As such, the easterly waves here could be a combination of many different modes of west-ward moving Rossby waves and mixed gravity waves that we may not be able to link them specifically to the equatorial Rossby waves. This has been now mentioned in this revision.

To what extent does this model reproduce the behavior of the traditional model for barotropic instability under parallel shear flow and the associated Rayleigh-Kuo conditions for instability? Based on Figure 2, it appears that the physical framework is the same. However, I am trying to understand the notion that the periodic state is a secondary stable state, which is in contrast to a traditional barotropic instability model in which the instabilities would be expected to continue to grow.

Our discussion in the previous version was indeed unclear, which may cause some confusion here. We would like to note that the periodic state is one of the stable branches of the Hopf bifurcation if the model parameter $R$ is slightly greater than the critical value. As long as $R$ is sufficiently close to the critical number $R^*$, this periodic state on the central manifold will maintain its stable structure. For a larger $R$ number, it should be noted that the stability of the periodic state may no longer be ensured, because the central manifold function must be re-evaluated and a complex structure of the model state may arise. To some extent, this is what anticipated in the real atmosphere, because not all easterly waves can become unstable and turn into TCs. Only under some certain condition do the easterly waves become unstable.

**Technical corrections:**

General: Suggest simply using “genesis” rather than “TCG” –acronyms are overused.
Thank you. We have tried to reduce the use of the TCG acronym as suggested.

P4L15: might note in the text for clarity that Delta here represents the Laplacian, which is typically denoted with nablaˆ2
Thank you. The notation nabla has been now defined explicitly.

P4L17: extra “the
Corrected.

P4L21: parenthesis error
Corrected.

P6L11: replace “while” with “though it”
Modified as suggested.

P5L28: “nonlinearity”
Corrected

P7L12: I believe this should be no v-wind component
You are right. We really meant $v$-wind here. This has been now corrected.

P8L15: “eigenvector”
The typo has been corrected

P9L13: “turns out”
Modified as suggested.

P10L19: “if it exists”
This sentence has been modified

P13L10: “all the same”
Modified as suggested.

P14L16: “nondimensional”
Corrected.

P14L20: issues with the parentheses
Corrected.

P14L24: “obtained from the”
The typo has been now corrected. We thank Reviewer 1 again for your various suggestions and comments.

**Response to Reviewer 2 (Dr. Alex Gonzalez)**

We wish to thank Reviewer 2 for your encouraging comments and very detailed corrections in the annotated supplement. In this revision, we have followed your suggestions and made all necessary changes. Below please find a list of the changes that we have made in response to your comments/suggestions in the annotated PDF.

This paper revisits the well-known topic of the ITCZ breaking down due to barotropic instability into individual vortices that can be seeds for tropical cyclones. This paper provides a new and unique perspective on the topic, showing extensive mathematical derivations about the zonal wavenumber that first becomes unstable in ITCZ breakdown and how this zonal wavenumber sets constraints on the size and total number of tropical cyclones on the globe at one time. Overall, the paper is well-written and the mathematical derivations appear to be accurate. The only concerns I have are about presentation quality. There are issues with the authors being too vague about the nomenclature in their derivations, which made it difficult to check all of the math. Additionally, the authors are not consistent in physically interpreting many of the parameters, which if fixed, would help more general audiences follow the entire paper. My concerns seem like they can be addressed relatively quickly, thus I am recommending acceptance of the paper after minor revisions.

Page 1, tile: Modified as suggested.

Page 1, line 5-6: Reworded as suggested.

Page 4, line 3: Yes, this is the trade wind. The phrase has been changed to directly indicate this.

Page 4, line 15: Thank you. The definition of the Laplacian operator has been added.

Page 4. This is the total derivative, as it contains the horizontal advection component as well, i.e., $\frac{d\Delta\psi}{dt} \equiv \frac{\partial(\Delta\psi)}{\partial t} - \frac{\partial\psi}{\partial y}\frac{\partial(\Delta\psi)}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial(\Delta\psi)}{\partial y}$ The existence of such horizontal advection is required such that the effects of background vorticity gradient can be properly taken into account as you commented (see the term that accounts for the background vorticity gradient in Eq. (8) for the contribution from the background $\widetilde{\psi_s}$.)

Page 5, table 1: Yes, the ITCZ latitude can be extended from 1500 km to 2000 km with little changes in our analyses or conclusion. This is because the meridional scale is still much less than the zonal scale, which is the circumference of the Earth (i.e., $2\pi R \sim 40{,}000\ km \gg 2000\ km$). Our choice of the ITCZ latitude around 12-15$^0$ is simply based on the typical latitude of the ITCZ during the peak TC season.

Page 5. The unit of Horizontal eddy viscosity coefficient should be “m^2 s^-1” has been corrected.

Page 5, line 6: The phrase has been added as suggested.

Page 5, line 11, 12: changed as suggested.

Page 6, line 12: Thank you. The typos related to the domain size has been now corrected.

Page 6, line 17: we have added the definition of the non-dimensional parameter $\gamma_1$ here. You are right, this is the ratio of the vorticity forcing and vorticity response.

Page 6, line 18: reworded as suggested.

Page 6, line 23: deleted as suggested.

Page 7, line 6: You are correct. $R$ is physically a ratio between the external forcing and the sum of the viscous and linear damping terms. This physical meaning has been now included in this revision.

Page 7, line 8: the typo has been corrected.

Page 7, line 12: Thank you. This is out typo. It should be no v-wind component, not u-wind component for the boundary condition here. This has been corrected.

Page 7, line 13: delete as suggested.

Page 7, line 15: delete as suggested.

Page 7, line 16: The physical meaning of three differential operators have been now added.

Page 7, line 22: Our typo here. It should be partial derivative here after expanding all of the terms.

Page 7, line 22: this is our typo. It should be minus sign here.

Page 8, line 16: reworded as suggested.

Page 8, line 17: The way we choose the zonal wave number $m$ is very similar to the way one solves, e.g., an oscillating string held fixed between two walls. Basically, the boundary condition dictates the possible values of the eigenstates. Given the periodic boundary condition in the zonal direction, the eigenvectors include therefore all possible eigenmodes $m, \forall m \in Z^+$. So, this differs from the Fourier expansion for which there exists a pair of dual amplitudes.

Page 8, line 18: Yes, $m$ represents zonal wavenumber or meridional mode. This has now been stated clearly here.

Page 8, line 19: reworded as suggested.

Page 8, line 21: reworded as suggested.

Page 9, line 1: reworded as suggested.

Page 9, line 2: $n$ represents the order of derivative w.r.t to $y$ direction. This has been now clearly indicated.

Page 9, line 9: We have moved the definition of $\phi_{m,n}$ to the above line to make it clearer.

Page 9, line 13: reworded as suggested.

Page 9, line 17: reworded as suggested.

Page 10, equation (34): Definition of the real part operator has been now included.

Page 10, line 18: deleted as suggested.

Page 10, line 19: modified as suggested.

Page 11, line 3: corrected.

Page 11, line 5: corrected.

Page 11, line 10: reworded as suggested.

Page 11, line 14: A separated equation has been added here. Note that different value of $m$ for different value of $L_y$ is what Fig 3 is about (the value of $L_y$ is encoded in the parameter $a$), and so we don't provide a separate table for this equation.

Page 11, line 17: Per your suggestion. we have now added some comment about the result in Neito Ferreira and Schubert (1997) of the most unstable zonal wavenumber being wavenumber 13 here.

Page 11, line 22: changed as suggested.

Page 11, line 23: reworded as suggested.

Page 11, line 25, 26: This is out typo. It should be 3000 km, not 3x3000 km. This has been fixed.

Page 11, line 27: Is there a citation to back up this statement? "that has been long observed but not fully explained so far."

Page 12, line 3: the Rayleigh and/or Fjørtoft necessary conditions for instability has been now explicitly mentioned in this revision.

Page 12, line 3: reworded as suggested.

Page 13, line 10: reworded as suggested.

Page 14, line 7: reworded as suggested.

Page 14, line 13: Thank you for commenting on this. Because $\nu\pi^4 \ll \alpha L^2\pi^2$ in our calculation of $R$ (see Eq 9), use of $\nu = 100, or\ 1000$ $m^2$ $s^{-1}$ would not change much our estimation of the critical number . We have now mentioned this explicitly in this work.

Page 14, line 28: turn "this new state" to "the disturbances in this new state".

Page 14, line 31: Thank you. The follow vectors have been added in this revision.

Page 15, line 8: changed to "feedbacks".

Page 15, line 11: changed to "discussed hereinafter".

Page 15, line 14: delete "," after (TCs).

Page 15, line 15: corrected as suggested.

Page 15, line 17: reworded as suggested.

Page 15, line 19: reworded as suggested.

Page 15, line 21: reference might help here "Using the Principle of Exchange of Stabilities condition for the ITCZ model"

Page 15, line 24: typo, add "." after generate.

Page 15, line 25: "k~12" = (the word estimation already implies an approximation).

Page 16, line 10: reworded as suggested.

Page 17, line 5: fixed.

Page 19, line 1: fixed.

Figure 3, 4: $m = 13$ has been added. The label axis has been also fixed.

Figure 5: Fixed.

**MANUSCRIPT TRACK-CHANGE**

```
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% Ocean Science (os)
% Primate Biology (pb)
% Proceedings of the International Association of Hydrological Sciences
(piahs)
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% Solid Earth (se)
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%\usepackage{tabularx}
%\usepackage{cancel}
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%\usepackage{float}
%\usepackage{subfig}
%\usepackage{rotating}


\begin{document}
\title{Large-scale Dynamics of Tropical Cyclone Formation Associated with
ITCZ Breakdown}

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% The [] brackets identify the author with the corresponding affiliation.
1, 2, 3, etc. should be inserted.
%
\runningtitle{Dynamics of Tropical Cyclone Formation}
\runningauthor{Kieu et al.}
\correspondence{ckieu@indiana.edu}
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**Deleted:** \Author[2]{Quan}{Wang}¶

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\maketitle
\begin{abstract}
This study examines the formation of tropical cyclones (TC) from the large-scale perspective. Using the nonlinear dynamical transition framework recently developed by Ma and Wang, it is shown that the large-scale formation of TCs can be understood as a result of the Principle of Exchange of Stabilities in the barotropic model for the Intertropical Convergence Zone (ITCZ). Analyses of the transition dynamics at the critical point reveal that the maximum number of TC disturbances that the Earth's tropical atmosphere can support at any instant of time has an upper bound of $\sim 12$ for current atmospheric conditions. Additional numerical estimation of the transition structure on the central manifold of the ITCZ model confirms this important finding, which offers an explanation for a fundamental question of why the Earth's atmosphere can support a limited number of TCs globally each year.
%, but also justifies the TC horizontal scale of $\sim 10^3$ km as observed.
\end{abstract}

\copyrightstatement{Reviewed manuscript}
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\section{introduction}
The life cycle of a tropical cyclone (TC) is typically divided into several stages including early genesis, tropical disturbance, tropical depression, tropical storm, hurricane, and finally the dissipation. Among these five stages, the tropical cyclogenesis (TCG), defined as a period during which a weak atmospheric disturbance grows into a mesoscale tropical depression with a close isobar and the maximum surface wind > 17 m s$^{-1}$ \citep{KaryampudiPierce2002, ToryMontgomery2006}, is perhaps the least understood due to its unorganized structure as well as ill-defined characteristics of TCs. During this genesis period (typically 2-5 days), synergetic interactions among various dynamical and thermodynamic processes at different scales can result in an eventually self-sustained, warm-core vortex before the subsequent intensification can take place. These early formation processes are so intricate that no single or distinct mechanism could operate for all TCs, rendering the genesis forecasting very challenging in practice. Such a multi-facet nature of TCG is the main factor preventing us from obtaining a complete understanding of TC formation and development at present.

Early studies by \cite{Gray1968, Gray1982} provided a list of necessary climatological conditions for TCG to occur, which include: (i) an underlying warm sea surface temperature (SST) of at least 26$^o$C; (ii) a finite-amplitude low-level cyclonic disturbance; (iii) weak vertical wind shear; (iv) a tropical upper tropospheric trough; and (v) a moist lower to middle troposphere. While the above conditions for genesis have been well documented in numerous observational and modeling studies since then, an elusive issue is that the actual number of TCs composes a small fraction of the cases that meet all these conditions in the tropical region every year. Moreover, TCG varies wildly among different ocean basin due to relative importance of large-scale disturbances, local

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forcings, and surface conditions, thus inheriting strong regional characteristics that common criteria may not be applied everywhere. For example, TC genesis in the North Atlantic basin often shows strong connection to active tropical waves originated from the South African Jet \citep{AvilaPasch1992, DeMaria1996, Molinari_etal1999}. In the northwestern Pacific basin, studies by \cite{Yanai1964, Gray1968, Gray1982, LanderHolland1993, RitchieHolland1997, Harr_etal1996, Nakato_etal2015} showed that the genesis is mostly related to Intertropical Convergence Zone (ITCZ) and monsoon activities. In the northeastern Pacific, vortex interaction associated with the topographic and tropical waves seems to generate abundant disturbances that act as the seeds of TC genesis \citep{Zehnder_etal1999, Molinari_etal1997, WangMagnusdottir2006, Halverson_etal2007, KieuZhang2010}.

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Other large-scale conditions that can interfere with TCG have been also reported in previous studies such as the Saharan air layer (SAL; \cite{DunionVelden2004}), upper-level potential vorticity (PV) anomalies \citep{MolinariVollaro2000, DavisBosart2003}, mixed Rossby-gravity waves \cite{AiyyerMolinari2003}, the ITCZ breakdown \citep{FerreiraSchubert1997, WangMagnusdottir2006}, or multiple vortex merges \citep{Simpson1997, RitchieHolland1997, WangMagnusdottir2006, KieuZhang2008, Kieu2015}. Along with this diverse nature of genesis in different basins, observational and modeling studies of TC development have shown that the evolution of tropical disturbances during the early genesis stage often encompasses a wide range of scales from convective-scale hot towers, mesoscale convective systems, to large-scale quasi-balanced lifting and cloud-radiation feedbacks \citep[e.g.,][]{RiehlMalkus1958, Yanai1964, Gray1968, ZhangBao1996a, RitchieHolland1997, Simpson1997}. In this regard, TCG is a truly multi-scale process that relative importance of different mechanisms must be carefully examined when studying the TC genesis in real atmospheric conditions.

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Recent effort in the TC genesis research has been shifted from examining local mechanisms to a broader perspective of how environmental conditions can produce and maintain TC disturbances during TC early development \citep{WangMagnusdottir2006, Dunkerton_etal2009, Montgomery_etal2010, Wangetal2012, Lussier_etal2013, Zhuetal2015, WuShen2016, Patricola_etal2018}. The most current attempt in quantifying the large-scale factors governing the genesis in the North Atlantic basin focuses on the so-called “pouch” conceptual model, which treats an early TC embryo as a protected region within large-scale easterly waves \citep{Wangetal2010, Wangetal2012, Dunkerton_etal2009, Montgomery_etal2010}. To some extent, this pouch idea can be considered as an advance of the requirement of an incipient disturbance for genesis to occur that was originally put forth by \cite{Gray1968}. Much of the development along this “pouch” idea has been on tracking wave packets in the co-moving frame required to protect the mid-level disturbances (the so-called Kelvin cat-eye in \cite{Dunkerton_etal2009, Lussier_etal2013}).

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Despite much progress over last decades, several outstanding issues in the TC genesis study still remain. From the global perspective, a particular question of what is the maximum number of TCs that the Earth's tropical atmosphere can form and support in any given day has not been

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adequately addressed. Answering this question will help explain a long-standing question of why the Earth has only a specific number of $\sim$ 100 TCs globally every year. A recent modeling study of the global TC formation by Kieu et al. (2018) demonstrated indeed that the daily number of genesis events is intriguingly bounded (<10), even in a perfect environment. This number is quite consistent with a simple scale analysis based on the typical scale of TCs with a diameter $\sim$ 3000-km, which shows that there should have less than 14 TCs on the Earth's atmosphere at any given time, assuming that the radius of the Earth is $\sim$6400 km. Using idealized simulations for a tropical channel, Kieu et al. (2018) showed in fact that genesis occurs in episodes of 7-10 storms each time with a frequency between the episodes of 12-16 days. This episodic development at the global scale as well as the upper bound of $\sim$ 10 storms for each episode as obtained from these idealized experiments suggests that there must have some large-scale environmental conditions or intrinsic properties of the tropical dynamics, which control the genesis processes beyond the basin-specific mechanisms as discussed in \cite{Patricola_etal2018} .

While recent advance in global numerical models can reasonably capture the very early stage of the TCG and serve as guidance for operational genesis forecasts, analytical models of TC development have been confined mostly to the later stage of TC development such that the axisymmetric characteristics of disturbances could be employed. The axisymmetry is critical for the theoretical purposes, because it reduces the Navier-Stokes equations to a set of approximated equations for which some balance constraints and simplifications can be employed.

Given the various basin-specific mechanisms that could produce TCs beyond the axisymmetric model for an individual TC, the main objective of this study is to focus specifically on a large-scale mechanism behind the formation of tropical disturbances associated with the ITCZ breakdown. This special pathway is very typical at the global scales whereby converging winds from the two hemispheres could set up a right environment for large-scale stability to develop \citep{Gray1968, Yanai1964, Zehnder_etal1999, Molinari_etal2000, FerreiraSchubert1997, WangMagnusdottir2006}. Indeed, satellite observations often show that the ITCZ tends to undulate and break into a series of mesoscale vorties or disturbances, some of which may eventually grow into TCs \citep{Agee1972, Hack_etal1989, FerreiraSchubert1997}. This is especially apparent in the WPAC basin where early studies by \cite{Gray1968, Gray1982} showed that TC genesis primarily occurs along the ITCZ, which accounts for nearly 80 percent of genesis occurrences in this area.

Although the ITCZ breakdown appears to be a slow process as compared to other pathways such as vortex merger \citep[e.g.,][]{WangMagnusdottir2006, KieuZhang2008, KieuZhang2010} or tropical easterly waves \citep[e.g.,][]{Zehnder_etal1999, Molinari_etal1997, Halverson_etal2007, Dunkerton_etal2009, Montgomery_etal2010, Wangetal2012}, it is an inherent property of the tropical atmosphere at the global scale that could provide a source of large-scale disturbances responsible for TCG. To minimize the complication due to the basin-specific features, we thus limit our study

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of the global TC formation to an idealized aqua-planet configuration to facilitate the analytical analyses in this study.

The rest of the paper is organized as follows. In the next section, an analytical model for the large-scale TC genesis based on the ITCZ breakdown model is presented. Section 3 presents detailed analyses of the principle of exchange of stabilities for the ITCZ model as well as the stability analyses of the dynamical transition. Numerical examination will be discussed in Section 4, and concluding remarks are given in the final section.

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\section{Formulation}
A unique characteristic of the ITCZ that provides a favorable environment for genesis to occur is the highly unstable zone along the ITCZ where trade winds from the two hemispheres converge. Such a zone with strong horizontal shear is well documented along the tropical belt where the potential vorticity gradient changes sign, providing a necessary condition for disturbances to grow according to Rayleigh's theorem \citep{CharneyStern1962, FerreiraSchubert1997}. Thus, a disturbance embedded within in the ITCZ can trigger a nonlinear growth and extract the energy from the background, resulting in a potential amplification of the disturbance.

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Because of such a dominant role of the ITCZ in the global TC formation, a natural model for TCG should take into account the large-scale ITCZ breakdown processes. This ITCZ breakdown model is particularly suitable for an aqua-planet that does not have other triggering mechanisms such as land-sea interaction or terrain effects. For this reason, we will consider the ITCZ breakdown as a starting model for the TC genesis at the global scale in this study. Inspired by the modeling studies of the ITCZ breakdown based on the shallow water equation by \cite{FerreiraSchubert1997}, we examine a similar model for the ITCZ dynamics on a horizontal plane in which the governing equation for the ITCZ can be reduced to an equation for the potential vorticity as follows

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\begin{align}\label{eq1}
\frac{d \Delta \psi}{dt}=\nu_{e}\Delta^{2}\psi +F-\alpha\Delta\psi-\beta
\frac{\partial\psi}{\partial x},
\end{align}
```

where the horizontal streamfunction $\psi$ has been introduced as a result of the continuity equation, $\nu_{e}$ is horizontal eddy viscosity, $\alpha$ is a relaxation time, $\Delta$ is the Laplacian operattor, and $F$ is an external force that represents the either a source/sink of mass within the ITCZ or vorticity source \footnote{In \cite{CharneyDeVore1979}, the relaxation time $\alpha$ is proportional to the ratio of the Ekman depth $D_E$ over the depth of the fluid $H$, while the external forcing term $F$ can be treated as a large-scale vorticity source.}. Note here that the derivative on the left hand side of Eq. \eqref{eq1} is the total derivative such that the horizontal advection of the vorticity is included. As discussed in \cite{FerreiraSchubert1997}, the mass source/sink term $F$ is important for the ITCZ dynamics, because the horizontal dynamics could not fully capture the vertical mass flux

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within the ITCZ. Unlike the original ITCZ model in
\cite{FerreiraSchubert1997}, we have however introduced in the above
model \eqref{eq1} an explicit drag forcing term to represent the impacts
of eddy diffusion as discussed in \cite[e.g.,][]{RambaldiMo1984,
LegrasGhil1985, FerreiraSchubert1997}. The governing equation \eqref{eq1}
for the horizontal streamfunction has been extensively used in previous
studies to examine the quasi-geostrophic dynamics under different large-
scale conditions \citep[e.g.,][]{CharneyDeVore1979, LegrasGhil1985,
RambaldiMo1984, Schar1990}.

To be specific for our TCG problem, we will apply Eq. \eqref{eq1} for a
zonally periodic tropical channel, which is defined as
\begin{align}\label{eq2}
\Omega=\left[0,L_{x}\right]
\times\left[0,L_{y}\right],
\end{align}
where $L_{y}$ is the width of the tropical channel in a hemisphere and
$L_{x}$ is the zonal length of the channel. This domain roughly
represents a region where the ITCZ can be treated as a long band wrapping
around the Equator. For the current Earth condition, $L_x \sim
40,000$ km, and $L_y \sim 1,000-1,500$ km (i.e., 10-15$^o$), and so by
definition $L_x \gg L_y$.
%
% Table 1
%
\begin{table}[t]
\caption{Parameters of the model}\label{t1}
\begin{tabular}{cll}
\tophline
$Variable$ & $Range$ & $Remark$\\
\middlehline
$U_o$ & 10-20 m s$^{-1}$ & Mean easterly flow in the tropical lower
troposphere \\
$L_y$ & 1200-1500 km & Width of the tropical channel $\Omega$ \\
$L_x$ & $\sim$40,000 km & Length of the tropical channel domain
$\Omega$\\
$a$  & $\frac{2L_y}{L_x}$ & Aspect ratio of the tropical channel \\
$\alpha$  & $ 10^{-5}-10^{-7} \mbox{s}^{-1}$ & Relaxation time \\
$\nu$  & $10-10^{4} \mbox{m}^{2} \mbox{s}^{-1}$ & Horizontal eddy
viscosity coefficient \\
$\beta$  & $2\times 10^{-11} \mbox{s}^{-1}$ & Variation of the Coriolis
parameter with latitudes \\
$\gamma$  & $10^{-10}-10^{-11} \mbox{s}^{-2}$ & Magnitude of the external
mass source/sink in the ITCZ breakdown model \\
\bottomhline
\end{tabular}
\end{table}

Before we can analyze the ITCZ breakdown model, it is necessary to have
first an explicit expression for the forcing term $F$. In the early study
by \cite{FerreiraSchubert1997}, $F$ represents a mass sink that is a
piecewise unit step function of latitudes. To account for the existence
of the zonal jet in mid-latitude regions, \cite{LegrasGhil1985} however
used a forcing of the form $F=\alpha \nabla \psi^*$, where $\psi^*$ is a
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given streamfunction that represents the zonal jet around $50^o N$. Given
our focus on the ITCZ dynamics, we will choose this forcing term such
that its corresponding steady state can best represent the typical
background flow in the tropical lower troposphere. A zonally symmetric
functional form for the $F$ that meets this requirement is
\begin{align}\label{eq2}
F=\gamma\sin\frac{\pi y}{L_{y}}
\end{align}
where $\gamma$ denotes the strength of the forcing. Note that this
forcing amplitude is not arbitrary, because its value dictates the zonal
mean flow in the tropical region as will be shown below.

While the forcing term given by Eq. \eqref{eq2} differs from the unit
step function in \cite{FerreiraSchubert1997}, it turns out that
\eqref{eq2} allows a steady solution consistent with the typical flow
near the ITCZ. Indeed, the steady state solution $\psi_S$ of \eqref{eq1}
that results from this zonally symmetric forcing is
\begin{align}\label{steady-state0}
\psi_{S}=\frac{-\gamma L_{y}^{4}}{\nu_{e}\pi^4+\alpha
L_{y}^{2}\pi^2}\sin\frac{\pi y}{L_{y}}.
\end{align}
The horizontal flow corresponding to this steady streamfunction is
illustrated in Fig. \ref{fig2}, which shows two opposite easterly and
westerly flows to the north and the south of an ITCZ during a typical
Northern Hemisphere summer as expected.

Given the above forcing $F$ and its corresponding steady state, the
problem of the ITCZ breakdown is now mathematically reduced to the study
of the instability of the steady-state \eqref{steady-state0} as the model
parameters such as the forcing amplitude $\gamma$, the relaxation time
$\alpha$, or the beta effect vary. To this end, it is more convenient to
re-write Eq. \eqref{eq1} in the non-dimensional form such that our
subsequent mathematical analyses can be simplified. Given the governing
equation \eqref{eq1}, it is apparent that the natural scaling for time,
streamfunction, and distance can be chosen respectively as follows:
\begin{align*}
t=\frac{1}{L_y \beta}t^*,~~
\psi=L_y U_{0}\psi^*,~~(x,y)=L_y(x^*,y^*),~~
F=\frac{\alpha U_{0}}{L_y}F^*,
\end{align*}
where the asterisk denotes the nondimensional variables, and $U_{0}$ is a
given characteristic horizontal velocity that determines the strength of
the zonal mean flow in the tropical region. Nondimensionalizing Eq.
\eqref{eq1} and neglecting the asterisks hereinafter, the nondimensional
form for Eq. \eqref{eq1} becomes
\begin{align}\label{eq3}
\frac{\partial\Delta \psi}{\partial t}+\epsilon
J(\psi,\Delta\psi)=&E\Delta^{2}\psi+
F-A\Delta\psi
-\frac{\partial\psi}{\partial x},
\end{align}
where
\begin{align*}
&\epsilon=\frac{U_{0}}{L_y^2\beta}~~ \text{is the Rossby number,} \\
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&E=\frac{\nu_{e}}{L_y^{3}\beta}~~ \text{is the Ekman number,}\\
&A=\frac{\alpha}{L_y\beta} ~~\text{is the ratio of the relaxation time to
the inherent time related to the Earth's rotation rate}.
\end{align*}

For the sake of mathematical convenience, we will hereinafter extend the
model domain from $[0,L_y]$ to $[-L_y,L_y]$ such that the boundary
conditions become meridionally symmetric along the Equator at $y=0$. This
mathematical method of extending the domain will simplify a lot of
calculations, though it has no effect on our solutions so long as we
limit the final solution in the original domain $[0,L_x]\times
[0,L_y]$ and maintain the Neumann boundary at $y=0$ as shown below. The
non-dimensional extended domain is therefore given by
\begin{align*}
\Omega=\left[0,\frac{2}{a}\right]
\times\left[-1,1\right].
\end{align*}
where the scale factor $a\equiv 2L_y/L_x$ is introduced to simplify our
spectral analyses. Given the above nondimensionlization, the non-
dimensional form of the forcing \eqref{eq2} is now simply
\begin{align}\label{eq4}
F=\gamma_{1}\sin\pi y,
\end{align}
where the nondimensional parameter $\quad\gamma_{1}=\frac{\gamma
L}{\alpha U_{0}}$ denotes the ratio of the vorticity forcing amplitude
$\gamma$ to the vorticity response $U_0$, and the non-dimensional form of
the steady state \eqref{steady-state0} is
\begin{align}\label{steady-state}
&\psi_{S}=-\frac{A\gamma_{1}}
{E\pi^4+A\pi^2}\sin\pi y.
\end{align}

We examine next the stability of the steady state \eqref{steady-state}
and how this critical point would bifurcate into new states as the model
parameters vary, using the dynamical transition framework developed by
\cite{MaWang2013}. To this goal, it is necessary to study the behaviors
of a deviation $\psi'$ around the given equilibrium \eqref{steady-
state0}. We follow the standard procedure in the dynamical transition and
expand the solution around the critical point \eqref{eq3} in the form
$\psi=\psi_{S}+\psi'$. It should be emphasized here that unlike the
traditional linear stability analyses in which one often assumes $\psi'
\ll \psi_S$, the dynamical transition framework does not impose such a
condition on $\psi'$. The sole purpose of introducing the partition
$\psi=\psi_{S}+\psi'$ is to simply shift the location of the stability
analyses towards the steady state $\psi_S$, much like shifting the
coordinate origin from $0$ to a new critical point in any linear
stability analyses. In Ma and Wang (2013)'s dynamical transition
framework, the full nonliterary is maintained such that possible analyses
of the central manifold can be carried out, and so no assumption of
$\psi' \ll \psi_S$ is needed in our analyses here. With this partition,
the corresponding governing equation for the perturbation $\psi'$ then
becomes
\begin{align}\label{eq5}
\begin{aligned}
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\frac{\partial\Delta \psi}{\partial t}+\epsilon
J(\psi,\Delta\psi)=&E\Delta^{2}\psi
-A\Delta\psi-\frac{\partial\psi}{\partial x}+
R\frac{d\widetilde{\psi}_S}{dy} \frac{\partial\Delta\psi}{\partial x}-
R\frac{d^3\widetilde{\psi_S}}{dy^3} \frac{\partial\psi}{\partial x},
\end{aligned}
\end{align}
where all the primes are hereinafter omitted for the sake of convenience, and a non-dimensional number $R$ and $\widetilde{\psi_S}$ are defined as follows
\begin{equation}\label{eq6}
R=\frac{\gamma_1\epsilon}{E\pi^3+A\pi},\qquad
\widetilde{\psi_S}=-\frac{\sin\pi y}{\pi}.
\end{equation}
Physically, the non-dimensional number $R$ is a ratio between the external forcing amplitude $\gamma_1$ and the sum of the viscous and linear damping terms. As will be shown below, this number turns out to be a key bifurcation parameter that determines the dynamical transition of the ITCZ breakdown model.

Given the nature of the ITCZ model, the periodic boundary conditions will be imposed in the zonal direction, and the free boundary conditions in the meridional direction for the perturbation equation \eqref{eq5} are applied at $y=-1$ and $y=1$ such that
\begin{align}\label{eq7}
\begin{aligned}
&\psi(t,0,y)=\psi\left(t,\frac{2}{a},y\right),\\
&\psi(t,x,-1)=\psi(t,x,1)
=0,\\
&\frac{\partial^{2}\psi}{\partial y^{2}}(t,x,-1)=\frac{\partial^{2}\psi}{\partial y^{2}}(t,x,1)=0.
\end{aligned}
\end{align}
The periodic boundary conditions along the west-east direction are naturally expected because of the cyclic property of the tropical channel around the Equator, while the free boundary conditions along the south-north direction will ensure that there is no meridional exchange (i.e., no $v$-wind component) at $y=-1$ and $y=1$. Apparently, the Neumann boundary condition at $y=0$ is still valid after the domain extension because of the continuity of the solution at $y=0$ in the interior region.

To further reduce the governing equation of the perturbation as given by Eq. \eqref{eq5}, we rewrite Eq. \eqref{eq5} in terms of an abstract functional notation that is standard in the study of the nonlinear dynamical transition. Introduce three differential operators $\mathcal{L},~\mathcal{G}$, and $\mathcal{A}$ as follows.
\begin{align}
\label{eq8a} &\mathcal{A}\psi \equiv \Delta\psi,\\
\label{eq8b} &\mathcal{L}\psi \equiv E\Delta^{2}\psi-A\Delta\psi-\frac{\partial\psi}{\partial x} + R\frac{d\widetilde{\psi_S}}{dy} \frac{\partial\Delta\psi} {\partial x}-R\frac{d^3\widetilde{\psi_S}}{dy^3} \frac{\partial\psi}{\partial x},\\
\label{eq8c} &\mathcal{G}\psi \equiv \epsilon J(\Delta\psi,\psi).

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\end{align}
Physically, $\mathcal{L}$ is the linear Laplacian operator,
$\mathcal{G}$ is the linear operator that contain the advection
associated with the background flow, and $\mathcal{A}$ is a nonlinear
operator representing the Jacobian effect. Eq. \eqref{eq5} for the
perturbation streamfunction with boundary condition \eqref{eq7} can be
then put into the following abstract operator form
\begin{equation} \label{eq9}
\frac{\partial \mathcal{A}\psi}{\partial t}=\mathcal{L}\psi-
\mathcal{G}(\psi).
\end{equation}
As seen in this abstract form, the operators $\mathcal{A}$ and
$\mathcal{L}$ are linear, whereas $\mathcal{G}$ is a nonlinear operator
due to the Jacobian's term. A standard procedure to Eq. \eqref{eq9} is to
employ the traditional bifurcation analyses and examine first the spectra
of the eigenvalues and eigenvectors of the linear component
$\mathcal{L}$. We then determine the stability characteristics of the
linear system, and finally construct the central manifold function with
the full nonlinear terms included so that the stable and/or unstable
properties of the new states of Eq. \eqref{eq5} can be quantified as the
model parameters vary. The outcomes from these analyses are i) the
conditions on the large-scale environment that could determine the
stability of the steady state as well as the upper bound on the number of
tropical unstable disturbances, and ii) the structure of new states after
the dynamical transition that the large-scale flows must possess to allow
for the formation of initial tropical disturbances. These outcomes are
interesting, because they could allow us to quantify the maximum number
of environmental tropical embryos that the ITCZ can support in the
tropical channel, thus addressing the question of how many TCs that we
would most expect in the tropical region at any given time.
%
% section
%
\section{An upper bound on unstable modes}
\subsection{Eigenmode analyses}
We start first with the search for the entire spectrum of the eigenvalues
$\rho$ of the linear operator $\mathcal{L}$  in \eqref{eq9}. Define a
linear operator $L(\rho)$ as follows
\begin{align}\label{eq10}
L(\rho)\psi&=\mathcal{L}\psi-\rho\psi,~~\rho\in \mathbb{C},
\end{align}
Then, all eigenvectors of the linear operator $\mathcal{L}$ are non-
trivial solutions of $L(\rho)\psi=0$ with a corresponding eigenvalue
$\rho$. Because of the periodic boundary condition in the $x$-direction,
it turns out that the eivenvectors cannot be arbitrary. Indeed, the
boundary conditions \eqref{eq7} impose a strict constraint on the
possible functional forms of $\psi$ such that every eigenvector $\psi$ of
$\mathcal{L}$ must be expressed in the following separable form
\begin{align}
\psi_m(x,y)=e^{i\pi max}\Psi(y),
\end{align}
where $m\in \mathbb{Z}$ is any integer representing the zonal eigenmodes,
and $\Psi(y)$ is the perturbation amplitude. Denote the corresponding
eigenvalue $\rho_{m}$ for each meridional mode $m$, a substitution of the
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preceding separable form into the eigenvalue equation
$L(\rho_{m})\psi=0$ yields
\begin{align}\label{eq11}
\begin{cases}
\mathcal{L}_{m}\Psi=\rho_{m}\mathcal{A}_{m}\Psi,\\
\Psi(-1)=\Psi(1)=\Psi''(-1)=\Psi''(1)=0,
\end{cases}
\end{align}
where each prime in Eq. \eqref{eq11} hereinafter denotes a derivative of
the streamfunction with respect to $y$, and the following notations have
been introduced:
\begin{align}
&\begin{aligned}\label{eq12}
\mathcal{L}_{m}\Psi\equiv&E(D^{2}-a^{2}m^{2}\pi^{2})^{2}\Psi-A(D^{2}-
a^{2}m^{2}\pi^{2})\Psi -ima\pi \Psi\\
&+ima\pi R\widetilde{\psi}_{S}' (D^{2}-a^{2}m^{2}\pi^{2})\Psi
-ima\pi R\widetilde{\psi}_{S}''' \Psi,
\end{aligned}\\
\label{ODE-eig2} &\mathcal{A}_{m}\Psi=(D^{2}-a^{2}m^{2}\pi^{2})\Psi,
\mbox{  and} \\
&D\equiv d/dy.
\end{align}
Applying the boundary conditions $\Psi(-1)=\Psi(1)=\Psi''(-
1)=\Psi''(1)=0$ to Eq. \eqref{eq11}, it can be seen that all even-order
derivatives of the perturbation amplitude $\Psi(y)$ vanish at the
boundaries, i.e.,
\begin{align}\label{eq13}
\Psi^{(2n)}(-1)=\Psi^{(2n)}(1)=0,\quad n=0,1,\cdots
\end{align}
where $n$ represents the order of derivative with respect to the
$y$ direction. This important property of the perturbation amplitude
$\Psi(y)$ results in a constraint that $\Psi(y)$ must be expressed in the
following form
\begin{align}\label{eq14}
\Psi(y)=\sum_{n\geq0}\phi_{n}\cos\left( n+\frac{1}{2}\right)\pi y
+\sum_{n\geq1}\widetilde{\phi}_{n}\sin n\pi y,
\end{align}
where $\phi_{n}$ and $\widetilde{\phi}_{n}$ are the coefficients to be
determined by the eigenvalue equation. As a result, every solution
$\psi_{m}(x,y)$ of $L(\rho)\psi=0$ can be expressed as
\begin{align}\label{eq15}
\psi_{m}(x,y)=\sum_{n\geq0}i^{n}e^{ima\pi
x}\phi_{m,n}\cos\left( n+\frac{1}{2}\right)\pi
y+\sum_{n\geq1}i^{n}e^{ima\pi x}\widetilde{\phi}_{m,n}\sin n\pi y,\quad
m\in \mathbb{Z},
\end{align}
where we have re-defined the expansion coefficients as
$i^{n}\phi_{m,n}$ and $i^n \widetilde{\phi}_{n}$  instead of
$\phi_{m,n}$ and $\widetilde{\phi}_{n}$ as in \eqref{eq14} for the sake
of convenience.

In what follows, we will determine the wavenumber $m$ such that the
eigenvector $\psi_{m}$ given by \eqref{eq15} becomes first unstable,
i.e., the real part of the corresponding eigenvalue $\rho_{m}$ becomes
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positive, as the control parameter $R$ increases. It can be verified that
for any complex eigenvalue $\rho_m \in \mathbb{C}$, $\psi_{m}$ and
$L(\rho)\psi_{m}$ will have the same functional form. Thus, let us denote
\begin{align}\label{eq16}
L(\rho_{m})\psi_{m}=&
L(\rho_{m})\sum_{n\geq0}i^{n}e^{ima\pi
x}\phi_{m,n}\cos\left( n+\frac{1}{2}\right)\pi
y+L(\rho_{m})\sum_{n\geq1}i^{n}e^{ima\pi x}\widetilde{\phi}_{m,n}\sin
n\pi y\\
\equiv &\sum_{n\geq0}i^{n}e^{ima\pi
x}\varphi_{m,n}\cos\left( n+\frac{1}{2}\right)\pi
y+\sum_{n\geq1}i^{n}e^{ima\pi x}\widetilde{\varphi}_{m,n}\sin n\pi y=0.
\end{align}
Apparently, \eqref{eq15} is an eigenvector of the eigenvalue equation
$L(\rho_{m})\psi=0$ if and only if the above identity is true $\forall
(x,y)$. As a result, explicit calculation of each term in Eq.
\eqref{eq16} will lead to
\begin{align}
\label{eq17a} &\varphi_{m,n}=B_{m,n+1}\phi_{m,n+1}+C_{m,n}\phi_{m,n}
-B_{m,n-1}\phi_{m,n-1}=0,~~n\geq1,\\
\label{eq17b}
&\varphi_{m,0}=B_{m,1}\phi_{m,1}+C_{m,0}\phi_{m,0}+i\left(A_{m,0}\phi_{m,
0}-
\phi_{m,0}\right)=0, ~~n=0,\\
\label{eq17c}
&\widetilde{\varphi}_{m,n}=D_{m,n+1}\widetilde{\phi}_{m,n+1}
+E_{m,n}\widetilde{\phi}_{m,n}-D_{m,n-1}\widetilde{\phi}_{m,n-1}=0,
~~n\geq 2\\
\label{eq17d}&\widetilde{\varphi}_{m,1}=D_{m,2}\widetilde{\phi}_{m,2}+
E_{m,1}\widetilde{\phi}_{m,1}=0, ~~n=1,
\end{align}
where the coefficients $A_{m,n}, B_{m,n}, C_{m,n}, D_{m,n}, E_{m,n}$ are
\begin{align}\label{relation}
&\begin{aligned}
&\begin{cases}
A_{m,n}=a^{2}m^{2}+\left(n+1/2\right)^{2}\\
B_{m,n+1}=(1-A_{m,n+1})\\
C_{m,n}=\frac{2\pi^{3}EA^{2}_{m,n}
+2\pi(A+\rho_{m})A_{m,n}-2iam}
{am\pi^{2} R}
\end{cases},\quad n\geq0,\quad |m|\geq1,\\
&\begin{cases}
B_{0,n+1}=(1-A_{0,n+1})\quad
A_{0,n}=\left(n+1/2\right)^{2}\\
C_{0,n}=2\pi^{3}EA^{2}_{0,n}+2\pi(A+\rho_{m})A_{0,n}
\end{cases}, \quad n\geq0,\quad m=0,
\end{aligned}
\\ \label{relation00}
&\begin{aligned}
&\begin{cases}
D_{m,n}=(1-\widetilde{A}_{m,n}),\quad
\widetilde{A}_{m,n}=a^{2}m^{2}+n^2\\
E_{m,n}=\frac{E\pi^3\widetilde{A}_{m,n}^{2}
+\pi(A+\rho_{m})
```

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\widetilde{A}_{m,n}-i2am}{am\pi^{2} R}
\end{cases},\quad n\geq1,\quad |m|\geq1,\\
&\begin{cases}
D_{0,n}=(1-\widetilde{A}_{0,n}),
\quad\widetilde{A}_{0,n}=n^2\\
E_{0,n}=E\pi^3\widetilde{A}_{0,n}^{2}
+\pi(A+\rho_{m})
\widetilde{A}_{0,n}
\end{cases}, \quad n\geq1,\quad m=0.
\end{aligned}
\end{align}
Given the conditions \eqref{eq17a}-\eqref{eq17d}, a simple way to obtain
the amplitudes $\phi_{m,n}$ and $\widetilde{\phi}_{m,n}$ is to group all
coefficients $\phi_{m,n}$ in each of these identities. This can be done
effectively by multiplying the conjugate coefficient
$\overline{\phi_{m,n}}$ and a factor $B_{m,n}$ on both sides of
\eqref{eq17a}-\eqref{eq17b}, and similarly
$\overline{\widetilde{\phi}_{m,n}}$ and a factor $D_{m,n}$ on both sides
of \eqref{eq17c}-\eqref{eq17d}. Adding the resulting identities together
and taking the sum over $n$ allows us to extract a relationship between
the amplitudes of $\phi_{m,n}$ and the eigenvalue $\rho_m$ as follows:
\begin{align}
\label{condition1}
&\sum_{n\geq0}B_{m,n}\varphi_{m,n}\overline{\phi_{m,n}}=0,\\
\label{condition2} &\sum_{n\geq1}D_{m,n}\widetilde{\varphi}_{m,n}
\overline{\widetilde{\phi}_{m,n}}=0.
\end{align}
Note that all pairs of the form
$(B_{m,n+1}B_{m,n}\overline{\phi}_{m,n+1}\phi_{m,n},
B_{m,n+1}B_{m,n}\phi_{m,n+1}\overline{\phi}_{m,n})$ in Eq.
\eqref{condition1} are conjugated to each other so that their sum will
produce a purely imaginary number. As a result, the real parts of
\eqref{condition1} and \eqref{condition2} must come from the term
involving $C_{m,n}$ and can be therefore obtained as
\begin{align}
\label{real-part} &\sum_{n\geq0}B_{m,n}A_{m,n}(E\pi^{2}
A_{m,n}+A+\Re[\rho_{m}])
|\phi_{m,n}|^{2}=0,\\
&\sum_{n\geq1}D_{m,n}
\label{real-part-2}
\widetilde{A}_{m,n}(E\pi^{2}\widetilde{A}_{m,n}+A+\Re[\rho_{m}])
|\widetilde{\phi}_{m,n}|^{2}=0
\end{align}
where $\Re$ denotes an operator of taking a real part of a complex
number. By imposing the physical requirement on the existence of the
eigenmodes with $\phi_{m,n} \ne 0$ and $\widetilde{\phi}_{m,n} \ne 0$,
Eqs. \eqref{real-part}-\eqref{real-part-2} can provide a great insight
into the stability and structure of the eigenmodes that we will now turn
into.
%
% subsection
%
\subsection{Upper bound on the unstable eigenmode}
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Eqs. \eqref{real-part}-\eqref{real-part-2} contain a number of very
powerful properties. First, note that the real part of the eigenvalue
$\rho_m$ for $m=0$ must be negative, if exist, due to the properties that
the coefficients $A>0$, $A_{0,n}>0$, $B_{0,n}\leq0$ and $D_{0,n}\leq 0$.
Indeed, if we assume that there exists an eigenvector $\psi_m$ such that
$\Re[{\rho_m}]>0$ for $m=0$, then it can be directly seen from the
quadratic form of \eqref{real-part} that
\begin{align*}
\phi_{0,n}=0,~~\widetilde{\phi}_{0,n}=0,\qquad n\geq0,
\end{align*}
and so there would exist no solution at all, which contradicts our
assumption of the existence of the eigenvector for $m=0$. Thus, the
zonally symmetric mode with $m=0$ is always stable. Because this stable
mode does not allow us to examine any transition behaviors, this special
mode will not be considered hereafter.

For $m\neq 0$, it can be seen also from \eqref{real-part} that all
possible unstable eigenvectors with $m\ne0$ must satisfy the following
constraints
\begin{align}\label{eq:constraint}
\begin{cases}
\begin{cases}
\phi_{m,n}=0,~~\text{when}, ~ a\geq\frac{\sqrt{3}}{2},~m\in Z;\\
\phi_{m,n}=0,~~\text{when},~ \frac{\sqrt{3}}{4}\leq
a<\frac{\sqrt{3}}{2},~|m|\geq2,\\
\phi_{m,n}=0,~~\text{when},~ \frac{\sqrt{3}}{6}\leq
a<\frac{\sqrt{3}}{4},~|m|\geq3,\\
\cdots\qquad\vdots\qquad\cdots\\
\phi_{m,n}=0,~~\text{when},~ \frac{\sqrt{3}}{2k}\leq
a<\frac{\sqrt{3}}{2k-2},~|m|\geq k
\end{cases}\\
\widetilde{\phi}_{m,n}=0,~~n\geq1, ~~\text{for all} \quad a>0.
\end{cases}
\end{align}
This conclusion can be explicitly confirmed if we note again that the
constraint \eqref{eq:constraint} will ensure that the coefficient
$A_{m,n}>0$, and $B_{m,n}<0$. If we assume that there exists any unstable
eigenvector $\psi_m$ with some wavenumber $m \ne 0$ such that the
corresponding eigenvalue $\rho_m$ satisfies $\Re[\rho_m]>0$, then Eq.
\eqref{real-part} immediately indicates that $\phi_{m,n}=0, \forall~ |m|
\ge k$ and $n\in\mathbb{Z}^{+}\cup\{0\}$ (i.e., $\psi_m=0$), and so no
such unstable eigenvector $\psi_m$ can exist at all. As a result, we
obtain a remarkable result that any possible unstable modes must be
constrained by the condition $|m|\le k$, where $k$ is an integer
satisfying the following relationship
\begin{equation}\label{maxm}
\frac{\sqrt{3}}{2k}\leq a<\frac{\sqrt{3}}{2k-2}.
\end{equation}

To help understand the significance of this result, we consider a
tropical channel domain between 10$^o$S-10$^o$N in the Earth's atmosphere
(i.e., $L_y \sim 1200$ km) and $L_x \sim 40,000$ km such that $a \equiv
2L_y/L_x \approx 0.06$. Using the condition $\frac{\sqrt{3}}{2k}\leq
a<\frac{\sqrt{3}}{2k-2}$, we obtain an upper bound wavenumber $k \approx
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12$. As can be seen from \eqref{maxm}, a narrower tropical channel width (i.e., smaller $L_y$) would lead to a smaller the scale ratio $a$, and so the upper bound $k$ will be higher. For our typical tropical region, the largest integer number $m \le 12$ is consistent with the most unstable zonal mode $m=13$ obtained from the modelling study in \cite{FerreiraSchubert1997}. %One could in principle choose the tropical width $L_y$ such that the upper bound $k$ perfectly matches with the most unstable zonal wavenumber 13 as reported in \cite{FerreiraSchubert1997}. Although we do not know exactly in advance which wavenumber $m < k$ will become unstable, because the condition $|m| < k$ includes a range of $m$ whose real part $\Re[\rho_m]$ could be positive, the above result is still very significant due to its explicit indication that the unstable wavenumbers cannot be arbitrary but must be bounded. Any eigenvectors with $|m| \geq k$ must be therefore stable and cannot grow.

%In this regard, this important finding indicates that the unstable modes that can grow and produce a favorable condition for TC genesis to occur are strongly constrained. %Due to the condition \eqref{eq:constraint}, all possible eigenvectors corresponding to unstable modes are therefore given by
%\begin{align}\label{eigv}
%\psi_{m}=\sum_{n\geq0}i^{n}e^{ima\pi x}\phi_{m,n}\cos\left( n+\frac{1}{2}\right)\pi y \mbox{ for } |m|\le k.
%\end{align}
A natural consequence of the above result is that not only the total number of TC disturbances has an upper limit, but the size of these disturbances must also be limited (i.e., the size of each disturbance is $\sim L_x/m$). If we assume that each of these disturbances could be eventually responsible for one TC embryo, the upper limit in the number of the disturbances as found from the above condition would imply a lower bound on the overall size of TCs, which has to be larger than $3000$ km in diameter. That is, the TC size on the Earth's atmosphere cannot be arbitrarily small, but must be larger than a limit of $\sim 10^3$ km, a fact that has been long observed but not fully explained so far. Of course, this TC size implication by no means eliminates the existence of a small TC such as midgets at the higher latitudes, because our analytical results basically provide only an estimate for the size of an area where a TC disturbance can emerge. Determining the actual size of a fully-developed TC requires, however, various complex factors beyond the scope of the TC genesis that is presented in this study \citep[e.g.,][]{Chavas etal2016}.

It should be stressed that, the condition on the unstable modes derived from the eigenvalue $\Re[\rho_m]$ as seen from \eqref{eq:constraint} is just a necessary condition, and it is by no means sufficient to specifically know which zonal wavenumber in the range of $[1,k]$ will become first unstable. Thus, we examine next how the real part of the eigenvalue $\Re{[\rho_m]}$ varies as the model parameter $R$ increases for each value of $m$. Note that the non-dimensional number $R$ encodes several important large-scale conditions including the Rossby number, the Ekman number, and the strength of the background flow $U_0$ as seen in \eqref{eq6}. As these large-scale conditions change, $R$ will vary as well. Depending on how the eigenvalue $\rho_m$ varies as a function of

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$R$, there may emerge a first unstable zonal wavenumber $m$ with a positive eigenvalue $\Re{[\rho_m]}$ that we need to quantify.

To ensure the existence of such a positive eigenvalue as $R$ increases, it is necessary to show that $\Re{[\rho_m]}$ must be an increasing function of $R$ such that the real part can become positive as $R$ increases, according to Fj{\o}rtoft's  theorem. The specific wavenumber $m$ for which $\Re{[\rho_m]}$ first becomes positive will possess the structure that dictates the new dynamical transition of the system, according to the Principle of Exchange of Stabilities (see Appendix 1 for the definition of this Principle). Due to the complication in deriving the exact expression for $\rho_m$, details of the derivations of $\Re{[\rho_m]}$ as a function of $R$ are provided in Appendix 2. An important conclusion from these derivations is that $\lim_{R\rightarrow+\infty}\Re[\rho_m(R)]=+\infty$, which then implies that there indeed exists a critical value $R^*$ at which $\Re{[\rho_m]}(R^*) =0$. This requirement is critical, since it directly indicates that the Principle of Exchange of Stabilities is ensured. More strictly speaking, this result means that there exists a positive integer $n\leq k$ and a critical Reynolds number $R^*>0$ such that the following conditions

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\begin{equation} \label{PES-1}
   \begin{cases}
  \Re [\rho_{n,1}] = \Re [\rho_{-n,1}]
  \begin{cases}
  >0 & \text{ if }  R> R^*,\\
  =0 & \text{ if }  R= R^*, \; \forall n=m_1,,\cdots,m_l \\
  <0 & \text{ if }  R< R^*,
  \end{cases},\\
  \Re [\rho_{m,k}] <0,~~\quad\text{ if } (m, k) \ne (m_{i},1),\quad1\leq i\leq l,\\
  \Im [\rho_{n,1}( R)] \neq0 \quad\text{for}~~  R\geq R^*,
\end{cases}
\end{equation}

must hold true. Corresponding to the first unstable mode $m$ and eigenvalue $\rho_{n,1}$, its eigenvector is then given by

\begin{align*}
\psi_{m}=\sum_{n=0}^{\infty}i^{n}e^{ima\pi x}\phi_{m,n}\cos\left( n+\frac{1}{2}\right)\pi y,~1\leq|m|\leq k.
\end{align*}

Note that these eigenvectors are unstable for $R > R^*$ and $|m|<k$ only, because all other eigenvectors $(|m|>k)$ are always stable as shown by the condition \eqref{eq:constraint}.

Due to the complicated expression for the eigenvalue $\rho_m(R)$ as shown in Appendix 2, the value $R^*$ cannot be exactly derived but must be numerically approximated for each $m$. The proof of $\lim_{R\rightarrow+\infty}\Re[\rho_m(R)]=+\infty$ provided in Appendix 2 ensures that $R^*$ always exists, and so it should be found numerically. Fig. \ref{fig3} shows the critical value $R_{m}^*$ as a function of $2/a$ for each value of $m$, which is obtained by using a numerical approximation. Note that for each value of $a$, there will exist only one value $k$ that satisfies $\frac{\sqrt{3}}{2k}\leq a<\frac{\sqrt{3}}{2k-2}$ and a value $m<k$ such that $\Re[\rho_{m,1}] = 0$. By searching for

the value of $R_m^*$ that ensures $\Re[\rho_{m,1}] = 0$, we obtain for each $m\leq k$ a curve $R_m^{*}=R_m^{*}(a)$ that determines the onset of the dynamical transition. Because the eigenvalues and the eigenfunctions corresponding to $-m$ are the complex conjugate of the respective eigenvalues and the eigenfunctions corresponding to $m$, only the cases of nonnegative $m$ need to be examined.

As shown in Figure \ref{fig3}, there are several key differences between the asymptotic limits of a very small and a very large $a$. Specifically, for a larger value of $a$ (i.e., a wider tropical region), the smaller wavenumbers $m$ will become unstable first, starting with $m=5$, which then decreases for a larger $R$. For the smaller value of $a$ (i.e., a narrower tropical channel), the larger wavenumbers will however become unstable first as shown in Figure \ref{fig3}. For example, for the typical scales of the Earth's tropical region with $L_x \approx 40,000$ km, and $L_y\approx 1,200$ km, $2/a = L_x/|L_y \approx 33.3 $. According to Fig. \ref{fig3}, the wavenumber $m=9$ will become unstable first as $R$ crosses the value $R^*=4$. Thus, the dynamical transition for $m=9$ will produce a new unstable wave structure corresponding to $m=9$ at the bifurcation point. As the parameter $R$ increases, other unstable modes corresponding to $m=8,7,6...$ start to emerge, thus producing more unstable structure as a result of the dynamical transition.

To focus on the wavenumber that is first unstable instead of the critical number $R^*$ as shown in Figure \ref{fig3}, Figure \ref{fig4} shows the first unstable mode $m$ as a function of $a$, assuming all of the same parameter values used in Figure \ref{fig3}. It can be seen in this Figure \ref{fig4} that for each value of $a$, there is only one wavenumber $n=n(a)$ for which $R^{*}_{n}=\min_{m\in\mathbb{N}} R^{*}_{m}$. This is the critical value $R^*=R_{n}^{*}$ at which the dynamical transition will occur according to the PES condition. In addition, one can better see how the first unstable mode depends on the aspect ratio of the tropical channel, with a higher wavenumber for a narrower tropical region. This same behavior is also valid for a range of values of the Ekman number $E$ and Rossby number $\epsilon$, which is not shown here because they do not provide any further information.

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% Section
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\section{Bifurcation structure}
While the analyses in the previous section could show an upper bound on possible unstable modes, the structure of the unstable modes as well as the subsequent effects of the nonlinear terms have not been discussed. Depending on the eigenvalues and the structure of the eigenvectors when the first dynamical transition takes place, one can reduce the full nonlinear system \eqref{eq9} to a central manifold and construct a central manifold function to examine the bifurcation and the structure of the new state with all nonlinear terms included. Standard procedure in dynamical transition \cite{MaWang2013} shows that once the steady-state $\psi_{S}$ loses its stability for $R>R^*$, the supercritical Hopf bifurcation may occur with a new stable state approximated as follows
\begin{align}\label{new-state}

\psi=\psi_S+\left( \frac{\Re(\rho_{n, 1})}{|\Re
(A)|}\right)^{\frac{1}{2}} f_n(x, y, t) + h.o.t.
\end{align}
assuming that the nondimensional parameter $R$ is sufficiently close to
$R^*$, i.e.,
\begin{align*}
0< \frac{R-R^*}{R^*}\ll1.
\end{align*}

Using a higher-order approximation around the critical point on the
extended central manifold, it can be shown that the manifold function
could provide a better approximation for $\psi$ when $R>R^*$ (Kieu et al.
2018). Nonetheless, the smooth behaviors of the eigenvector at
$R=R^*$ for the supercritical Hopf bifurcation suffices to indicate that
the structure of the solution at $R=R^*$ can well represent the behavior
of the new stable solution near $R=R^*$. Note that there may appear
either Hopf bifurcation or double Hopf bifurcation, depending on the
transition multiplicity at the critical value. This subtlety will
introduce much more complex analysis of the bifurcation and a transversal
intersection in the parameter plane that we will not present herein.

While these higher-order derivations of the central manifold function
require some technical details that are beyond the scope of this study
(see Kieu et al. 2018), it is possible to approach the bifurcation
structure problem by numerically solving the eigenvalue problem
\eqref{eq12}. Specifically, we notice that the $x$ dependence can be
obtained by simply searching for the first unstable mode $m$ as
$R$ approaches the critical value $R^*$. Using this numerical approach to
find the critical value of $R^*$, the entire spectrum of eigenvectors
associated with the potential new stable states after the dynamical
transition can be found for each set of large-scale environmental
parameters. We note at this point that the exact mode $m$ at which the
eigenvector becomes unstable is dependent on $R$ as shown in Figures
\ref{fig3} and \ref{fig4}. The only constraint we are certain of is that
$|m| < k$. Thus, the new stable mode for $R>R^*$ could inherit the
structure of any value of $|m|<k$ at $R=R^*$. This numerical approach is
powerful, as it allows one to search for not only the critical parameter
$R^*$ at which the PES condition is ensured, but also the structure of
new stable states for any value of $R>R^*$ after the bifurcation point.

To illustrate the results from this numerical approach, we assume the
following set of the large-scale environmental conditions in the typical
tropical region
\begin{align*}
L_y \sim1000\text{km},\quad U_{0}\sim 10\text{m s}^{-1},
\quad \alpha\sim10^{-6}\text{s}^{-1},\quad
\beta\sim10^{-11}\text{m}^{-1}\text{s}^{-1}, \quad \nu = 1000 \text{m}^2
\text{s}^{-1},
\end{align*}
which result in a Rossby number $\epsilon \approx 0.5$ and an Ekman
number $E\approx 0.05$. Further use of Eq. \eqref{steady-state0} for the
steady state and note that $U_0 = \partial \psi_S/\partial y$, one then
obtains also an estimation for the forcing amplitude $\gamma \approx
7\times10^{-10} s^{-2}$. From the definition of the nondimensional number

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$R$, we then get $R\approx 4.8$, which is above the critical value $R^* \approx 4$ for $m=9$ as shown in Figure \ref{fig3}. Thus, the PES condition is met, and a new stable structure must emerge after the dynamical transition. As a result, the eigenvalue problem \eqref{eq11} needs to be solved for the first eigenvector and its dual eigenvector, given this value for $R$. Note that this estimation of $R$ is most sensitive to the strength of the shear flow $U_0$, the beta effect $\beta$, and the scale $L_y$ but not on the eddy diffusion coefficient $\nu$. To some extent, this is what expected, because the large-scale eddy diffusion process is often negligible.

For this numerical method, we use a Legendre-Galerkin method where the unknown fields are expanded using a basis of $N$ polynomials, which are compact combinations of the Legendre polynomials satisfying the four boundary conditions \eqref{eq11} \citep{Shen2011} for the details of this numerical method). For the convergence of the numerical scheme, $N = 100$ is sufficient. Once the eigenvector problem is solved, a further approximation on the central manifold can be applied so that we can examine the stability of different states around the critical point on the central manifold.

Figure \ref{fig5} shows a new state as a result of the dynamical transition for $R> R^*$, which is obtained from the numerical procedures described above. Among several significant features of this numerical solution, the first noteworthy one is that the new state possesses a large-scale structure consistent with the ITCZ breakdown as observed in the idealized simulations by \cite{FerreiraSchubert1997}. Specifically, the tropical channel contains 10 large-scale disturbances, each has the horizontal scale of about 5000 km that could serve as embryos for the subsequent TC formation. Furthermore, the disturbances in this new state move to the left with a period of $T \sim 3.2116$ (i.e., $T \approx$ 4 days in the physical dimensional unit) as a result of nonzero imaginary part of the eigenvalues. It should be mentioned that the results shown in Figure \ref{fig5} hold for the Earth's tropical channel with $L_x/L_y \approx 36$ (or equivalently $a\approx 1/17$). For a different domain configuration such as different planets or climate with different tropical width, the unstable mode may be different and the maximum number of the TC-favorable disturbances will change as well.

That these large-scale structure of disturbances moving to the left with the temporal scale of $\sim$ 4 days as a consequence of the dynamical transition shown above is noteworthy, because these westward moving disturbances are to some extent similar to easterly waves in the real atmosphere. While it is entirely possible that these easterly waves are a mode of the equatorial mixed Rossby waves, it should be noted that the numerical procedure of finding the unstable mode on the central manifold presented in this section does not allow us to separate different modes of easterly waves. As such, these easterly waves could be a combination of different modes of west-ward moving Rossby waves and mixed gravity waves that we may not be able to be conclusive. In any case, the easterly waves that are often associated with the genesis can be now seen as a natural consequence of the dynamical transition, even for barotropic flows. Such consistency between the observed and theoretical estimation of the large-scale modes in the tropical region indicates that the

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barotropic instability and its inherent nonlinear dynamics can account for the pre-conditioning environment for TC genesis.

We should emphasize that the large-scale structure shown in Figure \ref{fig5} does not itself dictate that the disturbances have to grow and turn into TCs. Instead, these structures are simply new stable periodic solutions after the dynamical transition occurs. That is, for $R<R^*$, the stable structure is the steady state as given in Figure 2, whereas the new stable structure shown in Figure \ref{fig5} will emerge after $R>R^*$. As soon as these stable structures emerge, subsequent dynamic-thermodynamic feedbacks may be triggered, which result in further growth of the disturbances within each wave, similar to the pouch model proposed in Dunkerton et al. (2012). For the larger value of $R$, the stability of the periodic state may no longer be ensured, because the central manifold function must be re-evaluated and a new structure may arise. The subsequent intensification of any tropical disturbances as a result of the new unstable structure would require additional detailed physics that are, however, not the focus of this work, and so will not be discussed hereinafter. In this regard, the new stable periodic state shown in Figure \ref{fig5} serves only as a pre-conditioning environment for incipient disturbances to grow.

%While the new stable structure looks promising in explaining the maximum potential number of favorable environment for TC genesis to occur, we note that the behaviors of the system near the critical point are more subtle than the simple dynamical transition. This is because the dynamical transition depends on the multiplicity of the complex eigenvalues. As discussed in Kieu et al. (2018), the dynamical transition will be a continuous transition accompanied by the supercritical Hopf bifurcation or catastrophic transition accompanied by the subcritical Hopf bifurcation, depending on a transition number determined by the nonlinear interactions of the first eigenvector and its dual eigenvector. The situation will be even more complicated if the complex eigenvalue multiplicity is two. Regardless of the complexity in deriving the structure of the new steady state after the dynamical transition, we note that the new periodic state will be stable after the transition, which emerges as one of the possible modes $m<k$ such that the pre-conditioning environment for TC genesis to occur can be met.
%
% Section
%
\section{Conclusion}
In this study, we examined the dynamical mechanisms underlying the large-scale formation of tropical cyclones (TCs) using a barotropic model for the Intertropical Convergence Zone (ITCZ) driven by an external mass forcing. Assuming a variant type of a forcing that mimics the mass sink/source in the ITCZ as previously used in \cite{FerreiraSchubert1997}, it was shown that the large-scale steady flow (i.e., the critical point) in the ITCZ model loses its stability for a bounded range of the wavenumber $|m|<k$ if large-scale environmental conditions including the magnitude of the mean flow, the Ekman number, and the Rossby number satisfy certain constraints. That the number of the TC disturbances in the tropical region is upper bounded in any given day could offer an explanation for a fundamental question of

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why the Earth's atmosphere can support a limited number of TCs globally each year.

Using the Principle of Exchange of Stabilities condition for the ITCZ model, we found that the model undergoes a bifurcation and associated dynamical transition, which helps further determine the maximum number of TC disturbances that the Earth's atmosphere can generate. % by the condition $\frac{\sqrt{3}}{2k}\leq a<\frac{\sqrt{3}}{2k-2}$. Specifically, the theoretical estimation of the largest wavenumber $k$ that can still support the unstable structure as a result of the ITCZ breakdown is $k = 12$, assuming the typical characteristic of the Earth's tropical channel in which the zonal scale of the tropical channel is about order of magnitude larger than the width of the channel. Such a dynamical constraint on the maximum number of TC disturbanes is remarkable, as it suggest an intrinsic large-scale mechanism that controls the climatology of the TC numbers beyond the basin-specific features as recently noted in \cite{Patricola_etal2018}. Of interest, this constraint on the largest wavenumber of the unstable eigenmodes imposes not only an upper limit on the number of TC disturbances in the tropical region, but also results in a lower bound on the size of TC disturbances. This lower bound on the size of the tropical disturbances may help explain why TCs cannot be arbitrary small, but must be larger than a certain limit.

To verify our theoretical analyses, a numerical method is used to search for the structure on the central manifold of the ITCZ model as the model parameter $R$ is increased to a value larger than a critical value $R^*$. Here, the key parameter $R$ controlling the bifurcation in our ITCZ model is given by
\begin{align*}
R=\frac{\gamma\epsilon\pi}{E\pi^4+A\pi^2},
\end{align*}
where $\gamma$ is parameter measuring the strength of the ITCZ mass sink/source, $A$ is the parameter measure the effect of surface drag, $\epsilon$ is a parameter measuring the mean zonal flow, and $E$ is the Ekman number representing the eddy viscosity. Our numerical results confirmed that for $R>R^{*}$, a new large-scale state emerges whose structure depends on the value $R$. For $R$ sufficiently close to the critical value $R^*$, the new state possesses a new type of periodic motion with two groups of symmetric disturbances across the Equator. These new stable periodic solutions describe a type of westward-moving disturbances within the ITCZ, very similar to the classical easterly waves in the tropical region. The findings obtained from the ITCZ breakdown model in this study thus provide a new insight into the formation of TC disturbances in the Earth's tropical atmosphere, and provide a rigorous mathematical proof for the observation of the limited number of TCs annually at the global scale.

%Using numerical analyses, it was also found that as the parameter $R$ increases, there are one pair (or two pairs) of complex conjugate eigenvectors that becomes first critical. In the case with one pair of complex conjugate eigenvectors becoming first critical, the transition is either continuous or catastrophic as described by a Hopf bifurcation depending on the sign of a single non-dimensional transition number. For

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this case, our numerical results  showed that the system exhibits only super-critical Hopf bifurcations, i.e. there exists a stable time-periodic solution  associated with the mean flow \eqref{steady-state0}. For the case of two pairs of complex conjugate eigenvectors becoming first critical, there are many possible and interesting transition scenarios depending on the relations between four nondimensional transition parameters, which in turn depend on the nonlinear interactions of the critical modes with the stable modes. This case is non-generic and occurs when the control parameter crosses a co-dimension two critical point, which is only possible in multi parameter systems such as the one considered here. Such a critical point lies at the transversal intersection of two neutral stability curves of Hopf bifurcations and is known as double Hopf (or Hopf-Hopf) bifurcation in the bifurcation theory literature. In this case, our numerical experiments suggest that the system admits only one possible transition scenario and it is continuous type transitions as in the (single) Hopf case. Moreover, we prove that an $S^3$ -attractor bifurcates when this co-dimension two critical point is crossed in the parameter space. Under the restricted parameter regime dictated by our numerical observations, it is shown that the bifurcated attractor has a limit cycle or an invariant torus as a repellor.

%% The following commands are for the statements about the availability of data sets and/or software code corresponding to the manuscript.
%% It is strongly recommended to make use of these sections in case data sets and/or software code have been part of your research the article is based on.
%\codeavailability{TEXT} %% use this section when having only software code available
%\dataavailability{TEXT} %% use this section when having only data sets available
%\codedataavailability{TEXT} %% use this section when having data sets and software code available
%\sampleavailability{TEXT} %% use this section when having geoscientific samples available

\appendix
\section{Principle of Exchange of Stabilities}
The Principle of Exchange of Stabilities (PES) for a dynamical system basically refers to a critical condition for which the eigenvalues of the linear operator first cross a prescribed value. More precisely, the PES can be precisely stated as follows.

Let $\mathbf{L}_{\lambda}$ and $\mathbf{G}$ represent the linear and nonlinear parts of a dynamical system in the abstract form:
\begin{equation}\label{eq_a1}
\frac{d\mathbf{u}}{dt} = \mathbf{L}_{\lambda}(\mathbf{u}) + \mathbf{G}(\mathbf{u},\lambda)
\end{equation}
where ${\lambda} \in \mathbb{R}$ is the model parameter, and $\mathbf{u} \in \mathbb{R}^n$ represents the state of the system. By defniition, $\mathbf{L}_\lambda$ is a parameterized linear operator that depends continuously on $\lambda$. Consider the eigenvalue equation given by
\begin{equation}\label{eq_a2}
\mathbf{L}_{\lambda} \mathbf{e} = \beta(\lambda)\mathbf{e},

```
\end{equation}
where $\mathbf{e}$ is eigenvector, and $\beta(\lambda) \in
\mathbb{C}$ the eigenvalue. Let ${\beta_j (\lambda) \in \mathbb{C} | j
\in \mathbb{N}}$ be the eigenvalues (counting multiplicity) of
$\mathbf{L}_\lambda$. If we have
\begin{equation} \label{eq_a3}
   \begin{cases}
  \Re[\beta_j (\lambda)]
  \begin{cases}
  <0 & \text{ if } \lambda <\lambda_0,\\
  =0 & \text{ if } \lambda=\lambda_0, \; \forall 1\le i \le m\\
  >0 & \text{ if } \lambda<\lambda_0,
  \end{cases},\\
  \Re[\beta_j(\lambda_0)] <0, \; \forall j \ge m+1,
\end{cases}
\end{equation}
then the system is said to satisfy the PES condition at $\lambda_0$,
which signifies a bifurcation of the system from one state to another.
For dissipative systems, the PES condition has a much more powerful
implication than a simple bifurcation, as it ensures a dynamical
transition that can be completely categorized by three different types of
transition including the continuous transition, the catastrophic
transition, and the random transition. See \cite{MaWang2013} for more
details of the PES conditions for nonlinear systems.
%
% Appendix 2
%
\section{Existence of the critical number $R^*$}
For $\frac{\sqrt{3}}{2k+2}\leq a<\frac{\sqrt{3}}{2k}$ and $1\leq m\leq
k~(k=1,2,\cdots)$, it is easy to see from \eqref{real-part} that we must
have
\begin{align*}
\phi_{m,0}\neq0,
\end{align*}
because otherwise we will have $\phi_{m,n}=0,n\geq0$, and there would
exist no eigenvectors. For the sake of convenience, we will hereinafter
replace $\psi_{m}$ by $\frac{\psi_{m}}{B_{m,0}\phi_{m,0}}$, and similarly
replace $\phi_{m,n}$ by $\frac{\phi_{m,n}}{B_{m,0}\phi_{m,0}}$.
%Let's denote
%\begin{align*}
%\phi_{m,n}'=\frac{\phi_{m,n}}{B_{m,0}\phi_{m,0}},~n\geq0,
%\end{align*}
%and omitting primes,
Equations \eqref{eq17a}-\eqref{eq17b} are then re-written as follows:
\begin{align}\label{relation0}
\begin{aligned}
&B_{m,n+1}\phi_{m,n+1}+C_{m,n}\phi_{m,n}
-B_{m,n-1}\phi_{m,n-1}=0,~~n\geq1,\\
&B_{m,1}\phi_{m,1}+C_{m,0}\phi_{m,0}
+i\left(A_{m,0}\phi_{m,0}-
\phi_{m,0}\right)=0, ~~n=0,
\end{aligned}
\end{align}
and
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\begin{align}\label{real-part2}
&\sum_{n\geq0}B_{m,n}
A_{m,n}(E\pi^2A_{m,n}+A+\Re[\rho_m])
|\phi_{m,n}|^{2}=0.
\end{align}
Denote
\begin{align*}
d_{m,n}=\frac{C_{m,n}}{B_{m,n}},
\end{align*}
and let
\begin{align*}
\eta_{m,n}=B_{m,n}\phi_{m,n},
\end{align*}
\eqref{relation0} can be further rewritten as
\begin{align}\label{relation1}
\begin{aligned}
&\eta_{m,n+1}+d_{m,n}\eta_{m,n}
-\eta_{m,n-1}=0,\quad n\geq1,\\
&\eta_{m,1}+d_{m,0}
-i=0,\quad n\geq0.
\end{aligned}
\end{align}
This reduced equation \eqref{relation1} allows us to deduce a number of
important constraints. Indeed, we re-arrange \eqref{relation1} as
follows:
\begin{align}\label{relation2}
\begin{aligned}
&-d_{m,0}+i=\eta_{m,1},\quad\eta_{m,0}=1,\\
&\xi_{m,n}=\frac{\eta_{m,n}}{\eta_{m,n-1}}=\frac{1}
{d_{m,n}+\frac{\eta_{m,n+1}}{\eta_{m,n}}},\\
&-d_{m,0}+i=\frac{1}{d_{m,1}+\xi_{m,2}}=
\frac{1}{d_{m,1}+\frac{1}{d_{m,2}+\xi_{m,3}}}.
\end{aligned}
\end{align}
It is readily seen from \eqref{relation1} that $\eta_{m,n}=0$ for all
$n\geq0$ whenever there exists a $l \geq 0$ for which $\eta_{m,l}=0$.
This means that $\xi_{m,n}\neq0$ for all $n \geq 0$. From
\eqref{relation2} one can derive that
\begin{align}\label{expression}
\eta_{m,n}\equiv \xi_{m,1}
 \xi_{m,2}\cdots\xi_{m,n},~n\geq1.
\end{align}
Therefore, for $
\frac{\sqrt{3}}{2k+1}\leq a<\frac{\sqrt{3}}{2k}~(k=1,3,3,...)$,
\eqref{real-part2} can be equally rewritten as:
\begin{align}\label{equality}
\begin{cases}
\sum_{n\geq0}\Re [d_{m,n}]|\eta_{m,n}|^{2}=0,\\
\Re [d_{m,n}]<0(n\geq1), \Re[d_{m,0}]>0,
\end{cases}\quad
 m\leq k.
\end{align}
One can deduce from the third equality of \eqref{relation2} that
\begin{align}\label{eigenvalue11}
```

```
\rho_{m}=-A-\pi^{2}EA_{m,0}
+\frac{2iam+iam\pi^2 R\left(1-A_{m,0}\right)}{2\pi A_{m,0}}+\frac{\frac{-am\pi R\left(1-A_{m,0}\right)}{2A_{m,0}}}{d_{1}+\xi_{m,2}}.
\end{align}
Let's define a function $F$ using right hand side of
\eqref{eigenvalue11}, i.e.,
\begin{align*}
F(\rho_{m},R)=-A-\pi^{2}EA_{m,0}
+\frac{2iam+iam\pi^2 R\left(1-A_{m,0}\right)}{2\pi A_{m,0}}+\frac{\frac{-am\pi R\left(1-A_{m,0}\right)}{2A_{m,0}}}{d_{1}+\xi_{m,2}}.
\end{align*}
Due to the fact that
\begin{align*}
&\Re d_{m,n}<0(n\geq1),
\end{align*}
we can obtain that
\begin{align*}
|F(\rho_{m},R)|\leq&\left|-A-\pi^{2}EA_{m,0}
+\frac{2iam+iam\pi^3 R\left(1-A_{m,0}\right)}{2\pi
A_{m,0}}\right|\\&+\frac{\left|\frac{-am\pi
 R\left(1-A_{m,0}\right)}{2A_{m,0}}\right|}{|\Re [d_{1}]|}=K_{R}.
\end{align*}

\noindent Define a set $\Omega_{R}$ as
\begin{align*}
\Omega_R=\left\{z\in C\Bigg| \Re [z]>-A-E\pi^2\left(\frac{1}{4}+a^2\right),~|z|\leq K_R\right\},
\end{align*}
the Brown Fixed Point Theorem implies then that $F$ has a fixed point in
$\Omega_{R}$, i.e., there exists $\rho_{m}(R)$
such that
\begin{align*}
\rho_{m}(R)=F(\rho_{m}(R),R).
\end{align*}

At last, we prove that $\rho_{m}(R)$ is a continuous function of $R$ and
$\Im\rho_{m}(R)\neq0$.
Let $G(\rho_{m},R)$ be the function given by
\begin{align*}
G(\rho_{m},R)=F(\rho_{m},R)-\rho_{m}.
\end{align*}
If we can prove
\begin{align*}
\frac{\partial G}{\partial \rho_{m}}\neq0
\end{align*}
then the Implicit Function Theorem implies that $\rho_{m}(R)$ is indeed a
continuous function of $R$. From the definition of $G$ and
\eqref{eigenvalue11}, we obtain
\begin{align*}
\left|\frac{\partial G}{\partial \rho_{m}}\right|
&=\left|\sum_{n=1}(-1)^{n+1}\frac{\left(1-A_{m,0}\right)A_{m,n}}{A_{m,0}\left(1-A_{m,n}\right)}\eta_{m,n}^{2}(\rho_{m}(R))
-1\right|\\
&\geq1-\sum_{n=1}\frac{\left(1-A_{m,0}\right)A_{m,n}}
```

**Deleted:** $\rho_{m}(R)$ is indeed a continuous function of $R$.¶ Through \eqref{est1}, we can get that¶

```
{A_{m,0} \left(1-A_{m,n}\right)}|\eta_{m,n}|^{2}\\
&>1-\sum_{n=1}\frac{\left(1-A_{m,0}\right)A_{m,n}
\left(\Re[\rho_{m}(R)]+A+E\pi^2A_{m,n}\right)}{A_{m,0}
\left(\Re[\rho_{m}(R)]+A+E\pi^2A_{m,0}\right) \left(1-
A_{m,n}\right)}|\eta_{m,n}|^{2}\\
&=0.
\end{align*}
To prove $\Im[\rho_{m}(R)]\neq0$, we use the proof by contradiction.
Direct calculation gives
\begin{align*}
\frac{\left|\Im [d_{m,n}]\right|}{\left|\Re[ d_{m,n}]\right|}
=\frac{\left|2\pi \Im[\rho_{m}(R)] A_{m,n}-
2am\right|}{\left|2\pi^{3}EA^{2}_{m,n}
+2\pi(A+\Re[\rho_{m}(R)])A_{m,n}\right|}
\end{align*}
If $\Im\rho_{m}(R)=0$, we can deduce that
\begin{align*}
&\frac{\left|\Im [d_{m,n}]\right|}{\left|\Re [d_{m,n}]\right|}
=\frac{\left|2am\right|}{\left|2\pi^{3}EA^{2}_{m,n}
+2\pi(A+\Re[\rho_{m}(R)])A_{m,n}\right|}\\&>\frac{\left|2am\right|
}{\left|2\pi^{3}EA^{2}_{m,n+1}
+2\pi(A+\Re[\rho_{m}(R)])A_{m,n+1}\right|}=\frac{\left|\Im
[d_{m,n+1}]\right|}{\left|\Re [d_{m,n+1}]\right|},
\end{align*}
through which and combining the continuous fraction
\begin{align*}
-d_{m,0}+i=\frac{1}{d_{1}+\xi_{m,2}}=
\frac{1}{d_{m,1}+\frac{1}{d_{m,2}+\xi_{m,3}}}
\end{align*}
we get
\begin{align*}
\frac{\left|\Im [\eta_{m,1}]\right|}{\left|\Re
[\eta_{m,1}]\right|}<\frac{\left|\Im [d_{m,1}]\right|}{\left|\Re
[d_{m,1}]\right|},
\end{align*}
i.e.,
\begin{align*}
&\frac{\left|-\Im [d_{m,0}]+i\right|}{\left|-\Re
[d_{m,0}]\right|}<\frac{\left|\Im [d_{m,1}]\right|}{\left|\Re
[d_{m,1}]\right|}\Rightarrow\\
&\frac{\left|-\Im [d_{m,0}]+i\right|}{\left|\Im
[d_{m,1}]\right|}<\frac{\left|-\Re [d_{m,0}]\right|}{\left|\Re
[d_{m,1}]\right|}\Rightarrow\\
&1<\frac{\left|2am+1\right|}{\left|2am\right|}
<\frac{2\pi^{3}EA^{2}_{m,0}
+2\pi(A+\Re[\rho_{m}(R)])A_{m,0}}{2\pi^{3}EA^{2}_{m,1}
+2\pi(A+\Re[\rho_{m}(R)])A_{m,1}}<1,
\end{align*}
which leads to a contradiction. Hence, $\Im[\rho_{m}(R)]\neq0.$

\noappendix       %% use this to mark the end of the appendix section
\authorcontribution{CK perceived the model for TC genesis. QW carried out
mathematical proofs related to transition dynamics. AV conducted
```

numerical calculations. All authors contribute to the writing and the analyses of this work.}
\competinginterests{The authors declare that no competing interests are present}
\disclaimer{No}

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\end{acknowledgements}
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%REFERENCE 1
%\bibitem[AUTHOR(YEAR)]{LABEL2}
%REFERENCE 2
\end{thebibliography}
%
% FIGURES
%
% Figure 1
%
%\newpage
%\begin{figure}[p]
%\centerline{\includegraphics[scale=0.4]{figure1}}
%\caption{A time series of the number of TC genesis events detected each
day from an idealized simulation of TC formation in a tropical channel at
a homogeneous horizontal resolution of 27 km, using the Weather Research
and Forecasting (WRF-ARW) model (Kieu et al. 2018).}\label{fig1}
%\end{figure}
%
% Figure 2
%
\newpage
\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure2}\\
\caption{Illustration of the zonal wind that is derived from the steady-
state flow $\psi_{S}$ in the ITCZ model \eqref{eq1} with the external
forcing given by Eq. \eqref{eq2}. The dotted curve represents the
horizontal profile of the mean flow, while the black arrows represent the
direction of the mean flow for the tropical channel domain $\Omega_a$.
The blue dashed line denotes the location of the ITCZ.} \label{fig2}
\end{figure}
%
% Figure 3
%
\newpage
\begin{figure}
\centering
\includegraphics[width=1.0\textwidth,height=0.5\textwidth]{Figure1_r1}
\caption{Marginal stability curves $R_{m}^*(a)$ obtained from the
constraint on the eigenvalue $\Re\rho_{m,1}(R)=0$ for a range of the
aspect ratio $0.1\leq a\leq0.35$.} \label{fig3}
\end{figure}
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% Figure 4
%
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\begin{figure}
\centering
\includegraphics[width=1.0\textwidth,height=0.5\textwidth]{Figure2_r1}
\caption{The dependence of the first critical wave number $m=n$ on the
scale factor $a$, assuming the Rossby number $\epsilon = 0.5$ and the
Ekman number $E=0.05$ similar to Figure 2.}\label{fig4}
  \end{figure}
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% Figure 5
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\begin{figure}
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\includegraphics[width=1.1\textwidth]{Figure3_r1}\\
\caption{Illustration of the streamfunction $\psi$ for the new periodic
state on the central manifold near the critical point $R^*$ after the
dynamical transition, assuming $\epsilon=0.3$, $E=0.05$, and
$R=3.8717>R^*=3.8517$. The nondimenional period is T=2.776, which
corresponds to a physical period of $\sim $ 3.213 days. Superimposed are
corresponding vector flows derived from the streamfunction.}\label{fig5}
\end{figure}
%
% Figure 6
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%\newpage
%\begin{figure}
%  \centering
%  \includegraphics[width=0.9\textwidth]{figure6}\\
%  \caption{Sea level pressure (shaded) distribution as obtained from a
simulation of the tropical cyclone formation using the Weather Research
and Forecasting model for the tropical channel.}\label{fig6}
%\end{figure}
%\end{document}

%% Since the Copernicus LaTeX package includes the BibTeX style file
copernicus.bst,
%% authors experienced with BibTeX only have to include the following two
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%%
%% \bibliographystyle{copernicus}
%% \bibliography{example.bib}
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%%
%% URL = {http://www.xyz.org/~jones/idx_g.htm}
%% DOI = {10.5194/xyz}


%% LITERATURE CITATIONS
%%
%% command                        & example result
%% \citet{jones90}|               & Jones et al. (1990)
%% \citep{jones90}|               & (Jones et al., 1990)
%% \citep{jones90,jones93}|       & (Jones et al., 1990, 1993)
%% \citep[p.~32]{jones90}|        & (Jones et al., 1990, p.~32)
%% \citep[e.g.,][]{jones90}|      & (e.g., Jones et al., 1990)
%% \citep[e.g.,][p.~32]{jones90}| & (e.g., Jones et al., 1990, p.~32)
%% \citeauthor{jones90}|          & Jones et al.
%% \citeyear{jones90}|            & 1990
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%% FIGURES

%% When figures and tables are placed at the end of the MS (article in
one-column style), please add \clearpage
%% between bibliography and first table and/or figure as well as between
each table and/or figure.


%% ONE-COLUMN FIGURES

%%f
%\begin{figure}[t]
%\includegraphics[width=8.3cm]{FILE NAME}
%\caption{TEXT}
%\end{figure}
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%%% TWO-COLUMN FIGURES
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%\begin{figure*}[t]
%\includegraphics[width=12cm]{FILE NAME}
%\caption{TEXT}
%\end{figure*}
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%
%%% TABLES
%%%
%%% The different columns must be seperated with a & command and should
%%% end with \\ to identify the column brake.
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%%% ONE-COLUMN TABLE
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%\begin{table}[t]
%\caption{TEXT}
%\begin{tabular}{column = lcr}
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%\end{tabular}
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%\end{table}
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%%% LANDSCAPE TABLE
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%\end{tabular}
%\belowtable{} % Table Footnotes
%\end{sidewaystable*}
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%
%%% MATHEMATICAL EXPRESSIONS
%
%%% All papers typeset by Copernicus Publications follow the math
typesetting regulations
%%% given by the IUPAC Green Book (IUPAC: Quantities, Units and Symbols
in Physical Chemistry,
%%% 2nd Edn., Blackwell Science, available at:
http://old.iupac.org/publications/books/gbook/green_book_2ed.pdf, 1993).
%%%
%%% Physical quantities/variables are typeset in italic font (t for time,
T for Temperature)
%%% Indices which are not defined are typeset in italic font (x, y, z, a,
b, c)
%%% Items/objects which are defined are typeset in roman font (Car A, Car
B)
%%% Descriptions/specifications which are defined by itself are typeset
in roman font (abs, rel, ref, tot, net, ice)
%%% Abbreviations from 2 letters are typeset in roman font (RH, LAI)
%%% Vectors are identified in bold italic font using \vec{x}
%%% Matrices are identified in bold roman font
%%% Multiplication signs are typeset using the LaTeX commands \times (for
vector products, grids, and exponential notations) or \cdot
%%% The character * should not be applied as mutliplication sign
%
%
%%% EQUATIONS
%
%%% Single-row equation
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%\begin{equation}
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%\end{equation}
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%
%%% Multiline equation
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%\begin{align}
%& 3 + 5 = 8\\
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%%% MATRICES
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%\begin{matrix}
%x & y & z\\
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%%% ALGORITHM
%
%\begin{algorithm}
%\caption{...}
%\label{a1}
%\begin{algorithmic}
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%\end{algorithm}
%
%
%%% CHEMICAL FORMULAS AND REACTIONS
%
%%% For formulas embedded in the text, please use \chem{}
%
%%% The reaction environment creates labels including the letter R, i.e.
(R1), (R2), etc.
%
%\begin{reaction}
%%% \rightarrow should be used for normal (one-way) chemical reactions
%%% \rightleftharpoons should be used for equilibria
%%% \leftrightarrow should be used for resonance structures
%\end{reaction}
%
%
%%% PHYSICAL UNITS
%%%
%%% Please use \unit{} and apply the exponential notation


\end{document}
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