Atmos. Chem. Phys. Discuss.,
doi:10.5194/acp-2016-633-SC2, 2016

Atmospheric Chemistry and Physics
Discussions

Interactive comment on "The major stratospheric final warming in 2016: Dispersal of vortex air and termination of Arctic chemical ozone loss" *by* Gloria L. Manney and Zachary D. Lawrence

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Received and published: 27 September 2016

This document is intended for readers who are non-specialist in dynamical systems theory and who are using results on Lagragian Descriptors. The document attempts to clarify some key aspects of this methodology which have been misinterpreted in recent literature. I adopt the format of Frequently Asked Questions, in order to trace an easy to follow path for these readers.

1. What are Lagrangian Descriptors?

These are functions obtained from particle trajectories which evolve advected by fluid flows according to a dynamical system. These functions evaluate from time, $t-\tau$, to time, $t+\tau$, the integral along the particle trajectory of a *positive* quantity such as: mod-

ulus of the velocity, modulus of the acceleration, modulus of the velocity or acceleration raised to specific powers, etc. One of these functions very frequently used, called function M, is the one used by Manney and Lawrence in the work under discussion, that considers the integral of the modulus of the velocity along the trajectory. It provides the arclength of the path traced by the trajectory.

These functions are useful because they highlight, by means of singular features, invariant stable and unstable manifolds of hyperbolic trajectories of the underlying dynamical system. These mathematical objects are of geophysical interest because they are related to transport barriers of purely advected fluid particles.

2. What are singular features of the function M?

Singular features of the function M are described in Fig. 2 of the article by Mendoza and Mancho (Nonlin. Processes in Geophys. 2012) and in Figs. 10 and 11[1] of Mancho et al. (Comm. Non. Sci. Num. Sim. 2013) as abrupt changes in M which are quantified by discontinuities in the derivative of M along a specific direction crossing the manifold. The singular features in M are aligned with the invariant manifolds.

For discrete dynamical systems Lopesino et al. (Comm. Non. Sci. Nume. Sim. 2015) have defined a different kind of Lagrangian descriptor, maintaining the idea of integrating positive quantities along trajectories, but which allows a rigorous treatment. There singularities are discussed in terms of undefined derivatives of M at the points of the manifolds position, along lines which are transverse to the manifold.

3. What is novel in the method of Lagrangian Descriptors with respect to previous work based on time averages along trajectories?

Lagrangian Descriptors are based on integrals along trajectories. These can be converted into time averages by dividing by the time period of integration. Time averages have been related to phase space structures of dynamical systems through notions

[1] Fig. 11 has a typo in its caption. The caption should refer to Fig 10c) instead of 13c)

of ergodicity. A general framework that has been used in the context of fluid flows is the ergodic decomposition. This approach was developed in Malhotra et al. (Int. J. Bifurcation and Chaos 1998), Poje et al. (Phys Fluids, 1999), Mezic and Wiggins (Chaos 1999) and Susuki and Mezic (IEEE 2009) and is based on the fundamental work of Rokhlin (Am. Math. Soc. Transl. Ser. 1966). In particular Mezic and Wiggins (Chaos 1999) and Susuki and Mezic (IEEE 2009) highlight the importance of the Birkhoff ergodic theorem which states that in the limit $\tau \to \infty$ averages of functions along trajectories of measure preserving dynamical systems defined on compact sets do exist, and level sets of these limit functions are invariant sets. It should be noted that the Birkhoff ergodic theorem has *not* been proven for general velocity fields with aperiodic time dependence.

Results on Lagrangian Descriptors (LD) bring novel ideas with respect to these previous works:

• One novelty consists of the fact that LD are based on the integrals of *positive* quantities along trajectories forwards and backwards in time, while the work based on time averages considers the forward integration of *any* quantity along trajectories.

• A second novel aspect is that Mancho et al (2013) have shown that the integral of *certain positive quantities* (there exist positive quantities that fail in the goal) for sufficiently large integration time τ, highlight invariant manifolds of hyperbolic points by means of singular features which are visible in the function M. The method of time averages does not visualize the phase space structures by means of singular features, but by means of level sets once the average has converged.

• These differences make it possible for LDs to visualize the invariant manifolds of a simple linear saddle, while the method of the time averages cannot. The reason for that is that the averages along trajectories of typical reported quantities, such as the horizontal component of the velocity, does not converge (in fact the trajectories do not remain in a compact domain, as required), and thus level sets have no dynamical

interpretation.

• A third novel aspect is that singular features of LD are visible in time aperiodic dynamical systems such as those found in geophysical flows and accurately represent invariant manifolds as confirmed by numerical simulations contrasted with other techniques. The visualization of these features is accurate even if the average along the trajectories has not converged. We note that the Birkhoff ergodic theorem has not been generalized to the case of aperiodically time-dependent vector fields, and thus in these cases where LD provide insights the method of time averages discussed in the literature does not.

There exist cases, however, in which LD brings no novel aspect with respect to time averages. For instance the analysis of the dynamical behaviour around a linear elliptic point by means of LD does not highlight any singular feature aligned with invariant manifolds of hyperbolic points as there are no hyperbolic points in this case. Time averages of positive quantities along trajectories converge in this case because trajectories remain in a compact domain. In this particular example, once the average has converged, level sets are in 1-1 correspondence with the trajectories. Since LD are related to the time averages by a constant factor, level sets of LD computed for the required integration period, are also invariant sets.

4. What is the Objectivity property discussed in the literature? Is it important for practical purposes that LD satisfy that property?

A scalar valued, time-dependent, function is said to be objective if it is invariant under Galilean coordinate transformations. In other words, the pointwise values of a function are the same at points that are transformed under a Galilean transformation, for each value of time. The function M clearly does not have this property, and for this reason its utility for revealing phase space structure has been questioned in the literature (Haller, Ann. R. Fluid. Mech. 2015), although no examples of failure are pointed out.

The next example, however, shows that objectivity, understood as a property of functions which preserve pointwise values under a Galilean transformation, *is not a property that is desirable for any tool designed to reveal the phase space structure*, since the phase space structure may not be invariant under Galilean coordinate transformations.

For example, a system which is at rest, under a Galilean transformation having the form of a rotation with angular velocity $w = 1$, becomes described by the equations of a simple harmonic oscillator with both mass m and constant k equal to 1. The phase portrait of this system is described by concentric circles (1-tori) around an elliptic fixed point, and it is very different to the phase portrait of the system at rest consisting of a plane of fixed points. Therefore the M function should provide different information in each case –information that reflects the phase space structure for the particular dynamical system. In the answer to question 3 we have already explained how M recovers the phase portrait of a linear elliptic point. It is clear that if M satisfied the criterion of objectivity it would be the same for both systems and thus it would not distinguish between the phase space structure for each of these very different systems.

5. Do Ruiz-Herrera (Chaos 2015) results disqualify the use of Lagrangian Descriptors in Geophysical flows?

Ruiz-Herrera (Chaos 2015, arxiv 2015) provides a different approach to the concept of singularity in M to that referred in paragraph 2. He shows that in some specific examples in the limit $\tau \to \infty$, the function M has no singularities -in the sense that he has defined- that highlight invariant manifolds. Balibrea et al (arxiv 2015) show, however, that singular features -in the sense introduced in the paragraph 2- are still present in Ruiz-Herrera examples.

Ruiz-Herrera' results (Chaos 2015, arxiv 2015) are not applicable beyond the hypotheses satisfied by his examples which, on the other hand, as detailed below, are applicable to a rather limited type of flows. None of the geophysical flows used in the Manney and Lawrence work, under discussion here, are similar in any remote sense to

those assumed by the theorems in (Chaos 2015). Thus inferring that those theorems prove something about the velocity fields considered in this work is an unsupported statement. Furthermore the debate about the capacities of M for highlighting invariant manifolds in this open review manuscript is artificial simply because the authors use the function M with a different purpose.

Ruiz-Herrera' results are for particle trajectories mainly in unbounded 2D flows, in which at least one of the velocity components of the trajectory is unbounded and grows much faster than the other. Some of these assumptions are of crucial importance in his construction. These type of trajectories and flows, however, are rather far from those found in atmospheric and oceanic flows, in which particle velocities are bounded and oscillating, remain in finite domains and velocity fields are bounded and oscillating. More specifically, Ruiz-Herrera' theorem III.1 from (Chaos 2015) uses a linear velocity in the y-component, and Theorem III.2 uses a linear velocity in the x-component and y-component in some piecewise defined domains.

The debate introduced by Ruiz-Herrera on the inability of the M function to capture invariant manifolds and hyperbolic trajectories in the context of typical geophysical flows, solely based in his results, over exaggerates the scope of what he has actually done. Moreover, it contradicts plenty of published work in geophysical contexts showing numerical evidence which confirms the ability of M to capture invariant manifolds highlighted as singular features by systematically contrasting the method with other techniques such as direct computation of manifolds, advection of blobs, Finite Time Lyapunov Exponents, Finite Size Lyapunov Exponents, and observations of drifters and balloon trajectories:

1. C. Mendoza, A. M. Mancho. The hidden geometry of ocean flows. Physical Review Letters 105 (2010), 3, 038501-1-038501-4.

2. C. Mendoza, A.M. Mancho. The Lagrangian description of aperiodic flows: a case study of the Kuroshio Current. Nonlinear Processes in Geophsyics 19, (4) 449-472

(2012)

3. de la Cámara, A., Mancho, A. M., Ide, K., Serrano, E., and Mechoso, C. R.: Routes of Transport across the Antarctic Polar Vortex in the Southern Spring, Journal of the Atmospheric Sciences, 69, 741-752, 2012.

4. E. L. Rempel, A. C.-L. Chian, A. Brandenburg, P. R. Munoz, S. C. Shadden, Coherent structures and the saturation of a nonlinear dynamo, J. Fluid Mech. 729 (2013) 309-329.

5. A.M. Mancho, S. Wiggins, J. Curbelo, C. Mendoza. Lagrangian Descriptors: A Method for Revealing Phase Space Structures of General Time Dependent Dynamical Systems. Commun. Nonlinear Sci. Numer. Simul. 18, (12) 3330-3357 (2013)

6. de la Cámara, A., Mechoso, C. R., Mancho, A. M., Serrano, E., and Ide, K.: Isentropic Transport within the Antarctic Polar-Night Vortex: Rossby Wave Breaking Evidence and Lagrangian Structures, Journal of the Atmospheric Sciences, 70, 2982-3001, 2013

7. C. Mendoza, A. M. Mancho, S. Wiggins. Lagrangian Descriptors and the Assessment of the Predictive Capacity of Oceanic Data Sets. Nonlinear Processes in Geophysics 21, 677-689 (2014)

8. V.J. Garcia-Garrido, A. M. Mancho, S. Wiggins C. Mendoza. A dynamical systems approach to the surface search for debris associated with the disappearance of flight MH370. Nonlin. Processes Geophys., 22, 701-712, 2015 (2015)

9. A. Guha, C. R. Mechoso, C. S. Konor, R. P. Heikes, Modeling Rossby Wave Breaking in the Southern Spring Stratosphere. J. Atmos. Sci. 73 (1) 393-406 (2016).

10. V. J. García-Garrido, A. Ramos, A. M Mancho, J. Coca, S. Wiggins. A dynamical systems perspective for a Real-Time Response to a Marine Oil Spill. Marine Pollution Bulletin (2016). DOI: 10.1016/j.marpolbul.2016.08.018

Finally it is worthwhile to emphasize, specially for readers in the atmospheric sciences, that the fact that *observed* balloon trajectories in the lower stratosphere (Refs 3,6), or in situ oil spill *observations* in the Canary Islands (Ref. 10) follow the geometric structures extracted with LDs, provide strong empirical evidence that LDs are useful to study transport in realistic atmospheric/oceanic flows.

Please also note the supplement to this comment:
http://www.atmos-chem-phys-discuss.net/acp-2016-633/acp-2016-633-SC2-supplement.pdf

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Interactive comment on Atmos. Chem. Phys. Discuss., doi:10.5194/acp-2016-633, 2016.