The Method of Lagrangian Descriptors

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The M-function was used for the first time in the pioneering work

- A.J. Jimenez Madrid and A.M. Mancho, *Distinguished trajectories in time dependent vector fields,* **Chaos 19** (2009), 013111.

In two papers published in the same journal,

- A. Ruiz-Herrera, *Some examples related to the method of Lagrangian Descriptors,* **Chaos 25** (2015), 063112,

- A. Ruiz-Herrera, *Performance of Lagrangian Descriptors and Their Variants in Incompressible Flows*, **Chaos** (in press),

I have shown that the contours of the M-function have no implications in the detection of barriers to transport. Mancho, Wiggins, and their co-workers have posted on *arxiv* (without external peer review process) a paper that criticizes my work, namely

- F. Balibrea-Iniesta, J. Curbelo, V.J. Garcia-Garrido, C. Lopesino, A.M. Mancho, C. Mendoza, and S. Wiggins, *Response to: "Limitations of the Method of Lagrangian Descriptors"*[arXiv: 1510.04838]. arXiv preprint arXiv:1602.04243. (2016).

To defend their tool, they simply say that the method of Lagrangian Descriptors (LDs) does not involve the contours of the M-function. It is easy to observe that they misrepresent what they have actually done, (see comments below). We mention that Manney and Lawrence have detected the barriers to transport from the contour-lines of the M function.

Chaos: An Interdisciplinary Journal of Nonlinear Science offers the possibility of comments to regular papers, see *http://scitation.aip.org/content/aip/journal/chaos.* In fact, Mancho, Wiggins and their co-workers submitted a reply to *(Ruiz-Herrera 2015)* the last year. I strongly encourage to submit their new critique again in order to avoid misleading conclusions in the literature. (Limitation of the Method of Lagrangian Descriptors was the preliminary version of *(Ruiz-Herrera in press)*).

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1 Comments on "Lagrangian Descriptors for Two Dimensional, Area Preserving Autonomous and Nonautonomous Maps, Communications in Nonlinear Science and Numerical Simulation 27 (2015), 40–51 by C. Lopesino, F. Balibrea, S. Wiggins, A.M. Mancho"

In the aforementioned paper, Lopesino *et al.* have provided mathematical theorems to support the performance of the method of Lagrangian Descriptors in discrete systems.
Specifically, given $\{x_n, y_n\}_{n=-N}^{n=N}$ with $N \in \mathbb{N}$ an orbit of length $2N+1$ generated by a two-dimensional map, we consider

$$MD_p(x_0, y_0) = \sum_{i=-N}^{N-1} |x_{i+1} - x_i|^p + |y_{i+1} - y_i|^p \quad \text{with } 0 < p < 1. \tag{1.1}$$

They stated the following result:
Let

$$\begin{cases} x_{n+1} = \lambda x_n \\ y_{n+1} = \frac{1}{\lambda} y_n \end{cases} \tag{1.2}$$

with $\lambda > 1$.

Theorem 1.1. *Consider a vertical line perpendicular to the unstable manifold of the origin. In particular consider an arbitrary point $x = \bar{x}$ and a line parallel to the y axis passing through this point. Then the derivative of MD_p with $p < 1$, along this line becomes unbounded on the unstable manifold of the origin.*

The theorems presented in *(Lopesino et al. 2015)* are a consequence of the diagnostic itself because MD_p is non-smooth if, for some iteration,

$$x_{i+1} = x_i \;\; or \;\; y_{i+1} = y_i \tag{1.3}$$

Therefore, there are unbounded behaviours of the derivatives of MD_p in those points independently of the dynamical behaviour of the system. A more detailed discussion can be found in *(Ruiz-Herrera in press)*. As emphasized in my work, Theorem 1.1 gives no specific recipe for detecting unstable manifolds in general systems, it only works for systems satisfying (1.3). We also discussed in *(Ruiz-Herrera in press)* that Theorem 1.1 does not provide a mechanism to approximate the invariant manifolds when $N \longrightarrow \infty$ (this property was mentioned in Section 2.1.2 in (*Lopesino et al. 2015*)).

2 Critiques to the method of Lagrangian Descriptors in the literature

Apart from my work, many authors have criticized the method of Lagrangian Descriptors. Next, I give some examples:

Most applications of the M function have been published in *Nonlinear Processes in Geophysics.* A prerequisite for reliable predictions is the independence of the observer or objectivity. Unfortunately, as Haller nicely emphasized in

- *Non-objectivity of the M function and other thoughts*, Interactive discussion in Nonlinear Processes in Geophysics about the paper *Detecting and tracking eddies in oceanic flow fields by Rahel Vortmeyer-Kley, U. Grave, and U. Feudel.*

LDs are not objective. A.M. Mancho, as executive editor of that journal, provided a reply (without external evaluations) to Haller's comment. In her reply, she discussed the performance of the M-function in $x' = -y\ y' = x$. Of course, a concrete example is not sufficient to refute a general property. Moreover, she claimed: "the contours of M are in 1-1 correspondence with the trajectories of the system". There are several contradictions in this claim. First, the mentioned 1-1 correspondence is an exceptional feature of this particular system. For instance, it does not hold in $x' = x, y' = -y$. Importantly, her claim contradicts the fact exposed in *(Ballibrea-Iniesta et al 2016)* because they indicated that method of Lagrangian Descriptors does not involve the contour-lines of M.

In

- Fabregat, A., Mezic, I., and Poje, A. C. (2016). Finite-time Partitions for Lagrangian Structure Identification in Gulf Stream Eddy Transport. arXiv preprint arXiv:1606.07382.

we can find in page 16 after definition 3:

"The approaches based on integral of positive functions along a trajectory can fail to uniquely identify underlying objects. Specifically, the notion of Lagrangian Descriptors is based on this, but this notion is neither new, as it was proposed much earlier, nor complete."

3 Rebuttal to *(Balibrea-Iniesta et al. 2016)*

The method of LDs has been broadly applied by Mancho, Wiggins and their co-workers ($\sim$ 20 papers). They always analyze the contour-lines of the M-function (typically from data). For this reason, the statement of my results was on the behaviour of the contours of the M-function. However, they stated in their arxiv submission:

The term "singularity" does not refers to properties of the contour-lines of $M(x_0; t_0, \tau)$, but to points at which certain derivatives of $M(x_0; t_0, \tau)$ do not exist. While this is discussed somewhat in the references Mendoza and Mancho 2010 and Mancho et al 2013, it is made precise in Lopesino et al 2015

My response:

- As emphasized above, the conclusions of the theorems in *(Lopesino et al 2015)* are a consequence of the mathematical definition of the diagnostic. Therefore, those results have no dynamical significance. On the other hand, one can find in the introduction of that paper (page 41, line 10):

  Hyperbolic structures are revealed as **singular features of the contours of the lagrangian descriptors***, but the sharpness of these singular features depends on the particular norm chosen.*

  The analysis of the derivatives is carried out to extract information in the geometry of the contours of LDs. As emphasized in any paper of Mancho, Wiggins and their co-workers, the detection of barriers to transport always involves the contour lines of M. For instance, see figures 1-4 in *(Mendoza and Mancho 2010)*; figures 1-3, 5,7,9 in *(Camara et al 2013)* (the contour-lines of the derivative of M are missing); see figures of section 4 (applications) in *(Lopesino et al 2015)*; see figures 1-5 in *(Garcia-Garrido et al 2015)* and so on. In fact, in their last application *(Garcia-Garrido et al 2015)*, Section 4.1 says:

The structure of the M function shows, at low τ values, a smooth pattern such as that visible in Fig. 1a [...]. On the other hand, Fig 1.b (computed for $\tau = 20$ days) illustrate how the structures of M evolves for large τ towards less regular structures. By this we mean that sharp changes of M values occur in narrow gaps, forming filaments that highlight stable and unstable manifolds.

- The mathematical foundation of the method of LDs in continuous dynamical systems is given in *(Mancho et al 2013)*. In that paper, one can find the term "contour" exactly 68 times and expressions as

  The contours of M_1, for any fixed τ, are smooth circles surrounding the origin.

  As already noted the patterns of the contours displayed in Fig. 1 depends on τ. For τ small the contours are smooth but for increasing τ they develop singular features along the unstable and stable manifold.

  Singular contours of LDs correspond to invariant manifolds.

- In *(Balibrea-Iniesta et al 2016)*, the authors have used two outputs to capture barriers to transport in their systems. This is a contradiction with:

  The Lagrangian descriptors have the capability of revealing both the stable and unstable manifolds in the same calculation.

  See Section 6 in *(Mancho et al 2013)*. This property has been emphasized several times in *(Mendoza and Mancho 2010)*.

- Figure 1 in *(Ruiz-Herrera in press)* is exactly the same as Figure 1 (c) and 2(a) in *(Mancho et al 2013)*. However, they introduced the following misleading comment in *(Balibrea-Iniesta 2016)*, (see caption in figure 1):
  This figure should be compared with figure 1 of the comment of Ruiz-Herrera.

- The unbounded behaviour of the partial derivatives of M can not capture the dynamical behaviour of a dynamical system because M is typically unbounded. A more detailed discussion can be found in *(Ruiz-Herrera in press)*.