

# *Interactive comment on* "Quantifying the global atmospheric power budget" *by* A. M. Makarieva et al.

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Here we reply to Comments 3, 1 and 3 of, respectively, Referees 1, 2 and 4 addressing how the correct expression for atmospheric power W should look like. We note that the referees disagree on that matter. Referee 1 (and implicitly Referee 3) agree with our Eq. (7), which shows that W does not explicitly depend on the rate of phase transitions.

Meanwhile, Referees 2 and 4 opine, respectively, that our results either *appear to be* wrong or are unrelated to the standard view suggesting two derivations of their own. As shown below, both derivations assume that work per unit mass is equal to  $pd\alpha$ , which is not a valid assumption in the presence of phase transitions. The resulting expressions contradict not only our Eq. (7) but also the identical Eq. (4) of Pauluis and Held (2002) endorsed by Referee 3.

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The revised Section 2 below accounts for the above comments of the referees. It is followed by the comments themselves and our specific replies to them. Following the suggestion of Referee 1, we unburdened Section 2 of the longer derivation of Eq. (7) from the continuity equation and the ideal gas law and derived the same result immediately from the consideration of the relative change of the air parcel's volume. However, this simpler derivation contains an implicit assumption, which, as discussed in the revised text, necessitates our original longer derivation (in the revised text it is moved to the Appendix).<sup>1</sup>

# 2 Atmospheric power in the presence of phase transitions

When going from a dry to a moist atmosphere, where, besides dry air, there is also water vapor and the non-gaseous water (condensate) present, we need to accurately

$$W_{I} \equiv -\frac{1}{S} \int_{\mathcal{M}} \mathbf{v} \cdot \nabla p \alpha d\mathcal{M} = -\frac{1}{S} \int_{\mathcal{V}} \mathbf{v} \cdot \nabla p d\mathcal{V}, \qquad (1)$$

$$W_{II} \equiv -\frac{1}{S} \int_{\mathcal{V}} \mathbf{u} \cdot \nabla p d\mathcal{V}, \qquad (2)$$

$$W_{III} \equiv \frac{1}{S} \int_{\mathcal{V}} p \nabla \cdot \mathbf{v} d\mathcal{V}, \tag{3}$$

$$W_{IV} \equiv \frac{1}{S} \int_{\mathcal{M}} p \frac{d\alpha}{dt} d\mathcal{M}, \tag{4}$$

$$\frac{dX}{dt} \equiv \frac{\partial X}{\partial t} + \mathbf{v} \cdot \nabla X. \tag{5}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \dot{\rho},\tag{6}$$

<sup>&</sup>lt;sup>1</sup>This section follows Section 1, the relevant revised portion of which can be found in our first comment doi:10.5194/acp-2016-203-AC1. The relevant equations from the revised first section are

define velocity and density (Pelkowski and Frisius, 2011). We consider an atmosphere of total volume  $\mathcal{V}$  as composed of n macroscopic air parcels each of volume  $\tilde{V}_i$  (m<sup>3</sup>) such that  $\mathcal{V} \equiv \sum_{i=1}^{n} \tilde{V}_i = \int_{\mathcal{V}} d\mathcal{V}$ . Here  $\mathcal{V}$  can be defined as the volume bounded by the Earth's surface and the surface corresponding to some fixed pressure level  $p_T$  at the top of the atmosphere, e.g. to  $p_T = 0.1$  hPa. This is the uppermost level in many atmospheric datasets including those in the MERRA dataset. With  $\tilde{m}_d$ ,  $\tilde{m}_v$  and  $\tilde{m}_c$  being mass of, respectively, dry air, water vapor and condensate in a considered parcel, we define  $\rho \equiv \tilde{m}/\tilde{V}$  to be the air density,  $\tilde{m} \equiv \tilde{m}_d + \tilde{m}_v$ ,  $\rho_d \equiv \tilde{m}_d/\tilde{V}$ ,  $\rho_v \equiv \tilde{m}_v/\tilde{V}$ , and  $\rho_c \equiv \tilde{m}_c/\tilde{V}$  to be the condensate density.

Assuming the thermodynamic notion that work is performed due to expansion of a macroscopic body and is the product of pressure and volume change, the work of an air parcel per unit time per unit volume is

$$W_p \equiv \frac{p}{\tilde{V}} \frac{d\tilde{V}}{dt}.$$
(7)

At this point, as suggested by our Referee 1, it is tempting to note, following, for example, Batchelor (2000, p. 74), that the volume  $\tilde{V}$  of the air parcel changes as a result of movement of each element of the bounding material surface with velocity v, such that  $W_p$  becomes

$$W_p = \frac{p}{\tilde{V}} \int_{\tilde{S}} \mathbf{v} \cdot \mathbf{n} d\tilde{S} = \frac{p}{\tilde{V}} \int_{\tilde{V}} \nabla \cdot \mathbf{v} d\tilde{V} = p \nabla \cdot \mathbf{v}, \tag{8}$$

where n is the outward normal vector. The latter equality is valid in the limit of sufficiently small  $\tilde{V}$ . Then, the global atmospheric power per unit surface area can be defined and evaluated from the observed pressure and velocity of air as

$$W \equiv \frac{1}{\mathcal{S}} \sum_{i=1}^{n} W_{pi} \tilde{V}_{i} = \frac{1}{\mathcal{S}} \int_{\mathcal{V}} W_{p} d\mathcal{V} = \frac{1}{\mathcal{S}} \int_{\mathcal{V}} p \nabla \cdot \mathbf{v} d\mathcal{V}.$$
(9)

Eq. (9) is equivalent to Eq. (4) of Pauluis and Held (2002),  $W = W_{III}$ , see Eq. (3) above. Several essential comments are in order concerning the above derivation of W (9) (note that Pauluis and Held (2002) listed this equation without a derivation or reference). First, the above derivation considers the work of the expanding *air* that have pressure p. Hence, v in (9) is the velocity of the gaseous constituents of the atmosphere and not the mean velocity of gas and condensate.

Second, the derivation of Eq. (9) assumes the continuity of velocity at the parcels' boundaries, but does not assume the continuity of pressure. Indeed, in agreement with the thermodynamic definition of work (7), pressure is assumed to be the same everywhere within the air parcel although it can vary from parcel to parcel. (If one additionally requires the continuity of pressure (i.e. considers that pressure, too, varies across the parcel as velocity does) and defines work of a parcel not as  $p \int_{\tilde{S}} \mathbf{v} \cdot \mathbf{n} d\tilde{S}$  but as  $\int_{\tilde{S}} p\mathbf{v} \cdot \mathbf{n} d\tilde{S}$ , then the resulting expression for total work W would be  $(1/S) \int_{\mathcal{V}} \nabla \cdot (p\mathbf{v}) d\mathcal{V}$ , which is always zero.) For this reason, W (9) can be considered as the *definition* of macroscopic mechanical work per unit time (atmospheric power) that is consistent with the thermodynamic definition of work (7). Therefore, W (9) is a function of the temporal and spatial scale at which the macroscopic velocity  $\mathbf{v}$  is defined.

Third, quite remarkably, the above derivation of (9) appears to be invariant with respect to the presence or absence of phase transitions that change the amount of gas. That is, when deriving Eq. (9) we did not use the equality

$$W_p = \frac{p}{\tilde{V}} \left( \tilde{m} \frac{d\alpha}{dt} + \alpha \frac{d\tilde{m}}{dt} \right), \tag{10}$$

where  $\alpha \equiv 1/\rho = \tilde{V}/\tilde{m}$  is the volume occupied by unit air mass.

This is because Eq. (8) is based on an implicit assumption that the volume of the air parcel can change *only* when  $\nabla \cdot \mathbf{v} \neq 0$ , i.e. when the parcel boundaries move at different velocities *at the considered scale*.

Indeed, the standard thermodynamic interpretation of Eq. (9) is that if a certain parcel expands (positive work), the rest of the atmosphere contracts by the same amount (negative work), since the total atmospheric volume  $\mathcal{V}$  is constant. Thus, the expanding air parcels perform work on the compressing air parcels. When expansion and compression occur at different pressures, the resulting difference can be converted to mechanical work producing kinetic energy of wind which then dissipates to heat.

However, in the presence of phase transitions the situation is different. Consider for simplicity a parcel of water vapor in a still atmosphere composed of pure water vapor. Let it suddenly condense into a droplet. Despite the gas parcel has compressed into the negligible volume of the droplet, it did so *not because* some other parcel had expanded. In fact, no other air parcel performed any work on the condensing gas that suddenly reduced its volume. Instead, this reduction in volume  $d\tilde{V}/dt < 0$  occurred at the expense of the work of the intermolecular forces that ensured condensation. As condensation may occur nearly instantaneously (governed by velocities of the order of sound velocity), this type of volume change is generally *not accounted for* by the velocity divergence  $\nabla \cdot \mathbf{v}$  defined at an arbitrary scale. The question therefore arises whether the above derivation and the resulting expression  $W = W_{III}$  (9) can be generally reconciled with Eq. (10) for  $W_p$  in the presence of phase transitions.

We show in Appendix A that if we use the continuity equation and the ideal gas equation of state, the integration of  $W_p$  (10) yields Eq. (9). This is because the requirement of continuity postulates that any void space produced by condensation must be immediately filled by the expanding adjacent air parcels. This nearly instantaneous non-equilibrium expansion and the associated positive work of the air parcels precisely cancels the negative work of the intermolecular forces that make the air compress because of condensation. The requirement of continuity at a given spatial and temporal scale masks the two processes that occur at a different scale specified by the condensation process.

As we discuss in Section 6, since condensation is not spatially uniform, it is during

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such condensation-induced instantaneous expansion of the neigboring air parcels that the macroscopic pressure gradients may form to ultimately drive the atmospheric circulation and *determine* the value of atmospheric power W (9). In the conventional view the circulation arises when some air parcels aquire an opportunity to expand as they are receiving more heat (and thus warming more) than the others. The condensation-driven circulation arises when some air parcels acquire an opportunity to expand as the adjacent space is suddenly freed from the condensed gas. Notably, Eq. (9) does not carry information about the causes of circulation. It just gives the definition of macroscopic mechanical work per unit time (power) that is compatible with the thermodynamic definition (7).

Forth, Eq. (10) makes it clear that in the presence of phase transitions work done per unit mass is not equal to  $pd\alpha/dt$  (cf. Vallis, 2006, Eq. 1.65) but to

$$W_p \frac{\tilde{V}}{\tilde{m}} = W_p \alpha = p \left( \frac{d\alpha}{dt} + \frac{\alpha}{\tilde{m}} \frac{d\tilde{m}}{dt} \right) = p \left( \frac{d\alpha}{dt} + \alpha^2 \dot{\rho} \right), \tag{11}$$

where  $\dot{\rho} \equiv (d\tilde{m}/dt)/\tilde{V}$  (kg m<sup>-3</sup> s<sup>-1</sup>) is the source term from the continuity equation (6). It describes the local rate of phase transitions. The global integral of this additional term is not zero,  $\int_{\mathcal{M}} p\alpha^2 \dot{\rho} d\mathcal{M} = \int_{\mathcal{V}} p\alpha \dot{\rho} d\mathcal{V} \neq 0$ . Therefore, expression  $W_{IV}$  (4) that neglects this term is incorrect,  $W_{IV} \neq W_{III} = W$ . It cannot be used for evaluation of atmospheric power when the atmosphere has a water cycle.

Finally, we note that Eq. (9) does not assume stationarity. In the next section we consider how W can be decomposed into several terms with different physical meaning. This will clarify how  $W = W_{III}$  relates to  $W_I$  (1) and  $W_{II}$  (2).

### Appendix A. Deriving W (9) from $W_p$ (10) for ideal gas

The equation of state for ideal gas is

$$p = NRT. \tag{A1}$$

Here T is temperature, N is air molar density (mol m<sup>-3</sup>),  $V \equiv N^{-1}$  is the atmospheric volume occupied by one mole of air, p is air pressure and R = 8.3 J mol<sup>-1</sup> K<sup>-1</sup> is the universal gas constant.

Using (A1) we can now write  $W_p$  (10) as  $W_p = \frac{p}{\tilde{V}} \frac{d\tilde{V}}{dt} = \frac{p}{\tilde{V}} \left( \tilde{N} \frac{dV}{dt} + V \frac{d\tilde{N}}{dt} \right) = -\frac{dp}{dt} + RN \frac{dT}{dt} + \frac{RT}{\tilde{V}} \frac{d\tilde{N}}{dt}$ =  $RT \left( -\frac{dN}{dt} + \frac{1}{\tilde{V}} \frac{d\tilde{N}}{dt} \right)$ . Here  $\tilde{N}$  is the number of moles of gas within volume  $\tilde{V}, \tilde{V} = \tilde{N}V$ .

The number of molecules (moles)  $\tilde{N}$  in each air parcel can only change via an inflow (outflow) of molecules through the parcel's boundary. This change results from either diffusion of molecules between the adjacent parcels or from phase transitions or from both. Since in the case of diffusion any molecule leaving one parcel,  $d\tilde{N}_1/dt < 0$ , arrives to some other parcel,  $d\tilde{N}_2/dt = -d\tilde{N}_1/dt > 0$ , all the diffusion terms cancel in the global sum of the last term in Eq. (2) over all parcels. What remains corresponds to phase transitions:

$$\sum_{i=1}^{n} \frac{d\tilde{N}_{i}}{dt} = \int_{\mathcal{V}} \frac{1}{\tilde{V}} \frac{d\tilde{N}}{dt} d\mathcal{V} = \int_{\mathcal{V}} \dot{N} d\mathcal{V}, \tag{A2}$$

where  $\dot{N}$  is the molar rate of phase transitions per unit volume (mol m<sup>-3</sup> s<sup>-1</sup>). Its integral over volume  $\mathcal{V}$  is equal to the total rate of phase transitions in all the *n* air parcels. By virtue of the conservation relationship (A2)  $\dot{N}$  includes the inflow (outflow) into all the air parcels from all liquid or solid surfaces (droplet surface in the atmospheric interior or the Earth's surface).

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Using Eqs. (2) and (A2) we can write total power W of the n air parcels composing the atmosphere as

$$W \equiv \frac{1}{S} \int_{\mathcal{V}} W_p d\mathcal{V} = \frac{1}{S} \int_{\mathcal{V}} RT \left( \dot{N} - \frac{dN}{dt} \right) d\mathcal{V}.$$
 (A3)

Here  $dN/dt \equiv \partial N/\partial t + \mathbf{v} \cdot \nabla N$  is the material derivative of N with  $\mathbf{v}$  being the gas velocity.

On the other hand, the continuity equation can be written as

$$\dot{N} - \frac{dN}{dt} \equiv \dot{N} - \frac{\partial N}{\partial t} - \mathbf{v} \cdot \nabla N = N \nabla \cdot \mathbf{v}.$$
(A4)

Note that the left-hand side of the continuity equation is identical to the term in braces in Eq. (A3), the latter based on the equation of state (A1). Multiplying Eq. (A4) by RT and noting that p = NRT, we find that Eq. (A3) turns into Eq. (9).

The physical meaning of Eq. (A3) becomes clear from consideration of an atmosphere that is motionless on a large scale, such that  $\mathbf{v} = 0$  and  $\nabla \cdot \mathbf{v} = 0$ . Then condensation that occurs instantaneously on a smaller scale is described by the source term  $\dot{N} < 0$  that represents the large-scale mean. The compensatory expansion of the adjacent air is described by the large-scale mean  $\partial N/\partial t < 0$  showing that the molar concentration of air diminishes. As is clear from Eqs. (A3) and (A4), since  $\dot{N} - \partial N/\partial t = 0$ , no resulting work is performed on the considered scale: W = 0.

### Reply to the referees

We thank our referees for their comments.

Comment 3 of Referee 1 [doi:10.5194/acp-2016-203-RC1]: 3. The derivation of the total atmospheric power given by Eq. (7)-(8) is unnecessarily complicated. I can directly obtain Eq. (7) from Eq. (2) by noting simply that  $\int pdV/dt = \int pd(\delta x \delta y \delta z)/dt =$ 

 $\int p(\nabla \cdot v)dV$ . Not sure why the authors present their argument in such a lengthy and confusing way. The referee also noted in his general comments that the presentation of this work is somewhat confusing, and can be simplified substantially to make it clearer.

We followed the referee's suggestion to directly derive W from the definition of relative volume change. As we discuss in the revised Section 2, this derivation contains an implicit assumption that any volume change occurs at the expense of the divergence of velocity  $\nabla \cdot \mathbf{v}$  defined at an arbitrary scale. Since phase transitions involve gas velocities that are scale-specific, the plausibility of this assumption for this case requires a discussion, which is presented in the revised text.

Comment 2 of Referee 2 [doi:10.5194/acp-2016-203-RC2]: 1. Section 2 is both way too complicated and appears to be wrong. Following Vallis' (2006) notation:

$$W = \int_{\mathcal{V}} p \frac{d\alpha}{dt} \rho d\mathcal{V} = \int_{\mathcal{V}} p(\partial_t(\rho\alpha) + \nabla \cdot (\rho\alpha \mathbf{v}) - \alpha S_\rho) d\mathcal{V},$$

where  $S_{\rho} = \partial_t(\rho) + \nabla \cdot (\rho \mathbf{v})$  is the local sources and sinks of mass. Now,  $\alpha \rho = 1$  so

$$W = \int_{\mathcal{V}} p(\nabla \cdot (\mathbf{v}) - \alpha S_{\rho}) d\mathcal{V} = \int_{\mathcal{V}} (\nabla \cdot (p\mathbf{v}) - \mathbf{v} \cdot \nabla p - \alpha p S_{\rho}) d\mathcal{V}.$$

This is the same form as in equation (8). But it depends explicitly on  $S_{\rho}$ , contrary to the authors' claim. Why this contradiction? The problem in the authors' derivation comes in part from equation (3). While it is true that  $\sum_i d\tilde{N}_i/dt = 0$ , it is not true that  $\sum_i T_i d\tilde{N}_i/dt = 0$ , unless the atmosphere is isothermal. But it is exactly what's used to convert the last term in equation (2) to the last term in equation (4).  $\tilde{V} = \tilde{N}/N$  has units of  $m^3$  (parcel<sup>-1</sup>). To compute work, however, we need the specific volume with units of  $m^3$  (kg<sup>-1</sup>). So we have to introduce a new quantity, the mass per parcel  $\tilde{m}$  so that the specific volume is  $\tilde{V}/\tilde{m}$ . Then the expression for work (equation 4) with the same units as in Vallis (2006) reads:

$$W = \frac{1}{S} \int_{\mathcal{V}} p \frac{\tilde{m}}{\tilde{V}} \frac{d(\tilde{V}/\tilde{m})}{dt} d\mathcal{V}.$$
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But the continuity equation (6) also requires fixing. Since, *N* has units of mol  $m^{-3}$  then equation (6) is an equation for mass conservation only if the molar mass  $\tilde{m}/\tilde{N}$  is constant. But here the authors are, among other things, concerned about the effect of moisture on the work and moist air, unlike dry air, has an inhomogeneous in molar mass. The continuity equation (6) should then read:

$$\partial_t((\tilde{m}/\tilde{V})) + \nabla \cdot (\mathbf{v}(\tilde{m}/\tilde{V})) = \dot{\tilde{m}}(N/\tilde{N}) + \tilde{m}/\tilde{N}\dot{N} - (\tilde{m}N/\tilde{N}^2)\tilde{N}$$

where the right hand side is the local sources and sinks of mass. With these fixes, the expression for work will look exactly like in Vallis (2006) and will depend on the sources and sinks of mass.

The discrepancy between our Eq. (7) and the referee's derivation results from the incorrect definition of work. It is not  $pd\alpha$  as clarified in our revised text, see Eq. (11) above. Therefore,  $W \neq W_{IV} \equiv (1/S) \int_{\mathcal{V}} p(d\alpha/dt)\rho d\mathcal{V}$ .

We also note that in our derivation we did not assume either  $\sum_i d\tilde{N}_i/dt = 0$  or  $\sum_i RT d\tilde{N}_i/dt = 0$ . This misunderstanding might have arisen because the derivation was presented in a very compact form. The revised more detailed text (new Appendix A) makes it clear that the resulting expression for work does not depend on the temperature term discussed by the referee.

Comment 3 of Referee 4 [doi:10.5194/acp-2016-203-RC4]: Remarks on the methodology. Physically, the atmospheric energy budget is best understood by introducing some kind of available enthalpy  $ape = h(\eta, q_t, p) - h_r(\eta, q_t)$ , where h is the moist specific enthalpy,  $\eta$  is some suitable definition of moist specific entropy, and  $q_t$  the total specific humidity, p is pressure, where  $h_r(\eta, q_t)$  representing the part of the total enthalpy that is not available for adiabatic conversions into kinetic energy, so that

$$dh = (T - T_r)d\eta + (\mu - \mu_r)dq_t + \alpha dp$$

As a result, it is possible to express the total power term as

$$\int_{V} p \frac{D\alpha}{Dt} \rho dV = \underbrace{\int_{V} \frac{D(p\alpha)}{Dt} \rho dV}_{=0} - \int_{V} \alpha \frac{Dp}{Dt} \rho dV = \int_{V} \frac{T - T_{r}}{T} \dot{q} dm + \int_{V} (\mu - \mu_{r}) \frac{Dq_{t}}{Dt} dm$$

where  $\dot{q}$  represents diabatic heating terms by all manner of conduction of radiation. This neglects the integral of dh/dt, but this term could be retained if desired. The passage from the first term to the second term requires  $\nabla(\rho v) = 0$ , and  $\rho v$  to the total mass flux, in order to be able to claim that the integral of  $D(p\alpha)/Dt$  vanishes, so the authors should clarify this point, as well as boundary conditions assumed by the different velocities entering the definition of v. In any case, the above formalism is usually what constitutes the starting point for linking the atmospheric power budget to a Carnot-like theory and for constraining the atmospheric power budget to solar heating, sensible heat fluxes, and condensation/evaporation process. The approach proposed by the authors seem to be quite unrelated to this standard view.

The referee uses the same incorrect expression for work as Referee 2 above, with the same resulting discrepancies from our derivation. Total power is not equal to  $W_{IV} \equiv (1/S) \int_{\mathcal{V}} p(d\alpha/dt)\rho d\mathcal{V}$ , see Eq. (11) above. Moreover, since  $\nabla \cdot (\rho \mathbf{v}) = \dot{\rho} \neq 0$ , the second equation of the referee contradicts the first one.

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