Referee 3:

This manuscript attempts to view "atmospheric waves" as (scaling) turbulent phenomena by proposing the so-called turbulence-wave propagator with proper scaling. This study should be interesting to a broad audience and will have broad impact on research activities in both turbulence and atmospheric wave dynamics, which have been studied with different approaches separately, e.g., nonlinear vs. linear approaches. However, although a general term ``atmospheric waves" is used in this study, the mathematical foundations are based on the dispersion of the classical wave equation, which can only represent special kinds of atmospheric waves such as the Kelvin wave and 2D (x,y) gravity waves. Therefore, please be advised to construct the turbulence-wave propagator using a more general dispersion relation and then to examine if different atmospheric waves can be viewed as (scaling) turbulent phenomena. Alternatively, the authors may consider to revise the manuscript as well as its title to focus on the specific type of atmospheric wave (i.e., the Kelvin wave). In recognition and appreciation for the interesting and challenging study, the reviewer recommends that the manuscript be accepted for publication after the following issues are addressed. General and specific comments and a note on two linear wave equations with more general dispersion relations are provided below.

Au: Thank you for your positive evaluation. The reviewer's suggestion to somewhat widen the scope of the paper is a good one, which have followed to some extent. First, we have added a two new appendices. Appendix B gives the full general form of propagators satisfying scaling, causality and reality conditions and discusses the relations with group velocity and energy transfer. We have also explicitly related the dispersion relations and propagators to the more usual wave description in terms of travelling and evanescent waves. Finally – also as suggested - we have discussed the WKB approximation (appendix C). However, we have resisted the temptation to go much beyond this, in particular to attempt a re-evaluation of the many linear wave theories that the referee has kindly summarized in the second half of his comments. The reason is that the aim of the present paper was to use the concrete case of equatorial waves viewed by satellite in order to pose the question about the validity of linear wave theories. While it is obviously important to go beyond this case, it goes well beyond the framework of this paper, it is the subject of future work!

General Comments:

i. The authors are advised to use a more general dispersion relation (as discussed near the end of this comment file) to construct the turbulence-wave propagator.

ii. Two types of linear wave solutions are (1) propagating wave solutions in the form of $\exp(i\sigma t - imz)$ and (2) evanescent wave solutions in the form of $\exp(i\sigma t - Mz)$. Here $\boldsymbol{\sigma}$, m, and M are real numbers, and represent the frequency, wavenumber, and the reciprocal of the scale height, respectively. While the Kelvin wave propagates in the x direction, its meridional component (in the y direction) is evanescent. Thus, the "scaling" for the Kelvin wave is different from other types of waves which may propagate in both x and y directions (e.g., gravity waves or other types of waves). In addition, under certain atmospheric conditions (e.g., the occurrence of a critical level or critical latitude), the wavenumber in a specific direction may not be integers. Therefore, the existence of an isotropic scale really depends on type of wave. Several atmospheric waves with different dispersion relations are discussed in the note below.

Au: See the above comments and appendix B.

iii. A (linear) dispersion relation is derived from a set of linearized governing equations that describe the spatial distribution and temporal evolution of a flow with multiple fields (e.g., wind speeds, temperature etc) and the correlation among the different fields. The latter is often used to distinguish different weather systems. For example, the phase relationship between the low pressure center and convergence/divergence of wind fields is used to illustrate the differences between a mixed Rossby-gravity (MRG) wave and a tropical-depression- (TD-) type disturbance. Therefore, the reviewer is wondering if and how the concept of the fractional turbulence-wave propagator can provide something equivalent or similar (e.g., via a set of nonlinear governing equations) that can help identify different physical processes for different weather systems?

Au: The basic form and exponent of the fractional turbulence propagator is likely to be fairly fundamental (e.g. the H=1/3 in the Kolmogorov law is determined by dimensional considerations). However, the spatial scale function $\|\underline{k}\|$ may vary considerably from one realization to another and the relative importance of the wave and turbulent aspects (H_{wav} and H_{tur}) may similarly vary. This is a subject for future research.

iv. Linear wave theories have been developed to improve our understanding of the dynamics for terrain-induced or heating-induced mesoscale waves (e.g., Smith, 1979; Lin 1987; Lin 2007 and references therein), and for large-scale equatorial tropical waves (e.g., Matsuno 1996; Wheeler and Kiladis 1999). These studies have suggested an effective means of detecting atmospheric waves in real data. By analyzing global analysis data, Frank and Roundy (2006) and Schreck et al. (2012) have shown the strong relationship between tropical wave activities and tropical cyclone (TC) genesis. These tropical waves may be viewed as precursors to TC genesis. The association of TC genesis with different (linear) tropical waves has been illustrated with modeling studies (Shen et al., 2012; 2013 and references therein), leading to the hypothesis that the lead time of TC genesis prediction can be extended by improving the representation and evolution of (linear) tropical waves and their modulations on TC activities. In addition, the linear wave solutions have been used to verify the solutions of the numerical models. The aforementioned studies, just to name a few, have illustrated the usefulness of linear wave theories (for short-term weather simulations, at least). As the authors conclude that no linear theories are needed, it is important for the authors to provide strong justifications for the superiority of the proposed approach (with the turbulence wave propagator) with respect to the linear wave approaches.

Au: The scaling propagator approach is superior since it does not make unrealistic assumptions about the dynamics in order to linearize them. Obviously, it will need to be applied to the problem of TC genesis before it can be shown to be of practical value in *that situation. At the moment, in as much as linear wave theories mostly rely on dispersion relations, we mostly claim that we expect to be able to reproduce results consistent with linear wave theories (nearly the same dispersion relations). One difference between the approaches is in the exponents determining the fall-off of wave amplitudes – if more realistic - this difference could represent a practical improvement of our approach.*

Specific Comments

page 14798, line 18-19: please consider to expand the discussions on the linear atmospheric waves. Some references are provided below.

page 14800, line 17-18: The authors state `` A key characteristic of linear theories is that they involve integer powers of the (space and time) differential operators''. When atmospheric conditions are not "uniform," the governing equation such as Eq. (2.71) in the section 2.3 of Nappo (2002) can be solved only by the so-called W.K.B. method. Therefore, is the above statement still valid?

Au: Thank you. We have added a an entire appendix on the WKB method which combines both a theoretical and empirical assessment in the classical case of gravity waves.

page 14801-14802, Eqs. (2) and (4): When other dispersion relations are used, can you always find the corresponding fractional propagators? if so, will the "anomalous" exponent H be the same or different? Is the value of H dependent of data type and length?

Au: The general case is now given in appendix B, the classical wave equation was only used as a motivational example. The overall scaling exponent $H=H_{tur}+H_{wav}$ is presumably basic and depends on the phenomena, however, it is quite possible that the individual values of H_{tur} and H_{wav} depend on the meteorological situation (they are random variables).

page 14805, line 25: please provide justifications for the scaling: $(k_x,k_y,\omega) \rightarrow \lambda(k_x,k_y,\omega)$ for the Kelvin wave (whose meridional component is evanescent) and the other types of waves as well.

Au: The justification for the overall scaling $(k_x,k_y,\omega) \rightarrow \lambda(k_x,k_y,\omega)$ is empirical, it is from fig. 1. The proposed dispersion relation satisfies all the criterion (including scaling) as discussed in detail in appendix B.

page 14806-14807, line 25: The authors state: ``Although the dispersion relation is independent of the propagator exponent H_{wav}; the exponent does determine the (power law) rate of decay of the forcing so that the value of H_{wav} will affect the transport of momentum and energy." The reviewer has two questions:

(1) when different dispersion relations are used, is the above statement still valid?

Au: Yes, see the results in the new appendix B.

(2) Given a specific H_{wav}, how can the transport of momentum and energy be determined? Namely, what are the mathematical expressions for the transport of momentum and energy?

Au: See appendix B.

page 14807, line 10-12: as $\left\|\underline{k}\right\| = \left(k_x^2 - a^2 k_y^2\right)^{1/2}$ is used, which is consistent with the characteristic of the Kelvin wave, the authors may want to make changes in Eq. (14) on page 14805.

Au: the text has been modified accordingly.

page 14808, line 9: ``Data were divided into five 277h (~12 day) blocks, and each block is
calculated …'' please provide justifications for the choice of 277h? In addition, is the spectral
density analysis sensitive to the choice of this time scale?

Au: We added the following to the text:
"The choice of 12 day blocks was made since the temporal scaling has a break at about 5 - 10 days and we were only interested in analyzing the high frequency "weather" regime. Choosing a longer block period would allow us to examine lower frequencies, but would take us outside the unique scaling regime consider in the paper and would decrease the number of blocks and hence the amount of averaging."

page 14809, line 1: $H= H_{tur}+H_{wav}$ indicates that the total propagator exponent (H) is a linear superposition of H_{tur} and H_{wav}. Therefore, given the dispersion relation in this manuscript, the ``dynamics'' (e.g., energy transfer) of the turbulence and waves depend on H_{tur} and H_{wav}, respectively. However, how can this concept be generalized when other waves with dispersion relations are considered?

Au: The equation $H= H_{tur}+H_{wav}$ is actually a consequence of the assumption of a multiplicative structure of the propagator (following Wheeler and Kiladis). As noted via an addition of a new comment, it is plausible since it means that the space-time localized turbulent flux source φ *can be replaced by a "smeared out" turbulent source* $\varphi' = g_{tur} * \varphi$. *The amplitudes of propagation due to the wave part are indicated in appendix B.*

page 14809, Figure 3: was only the Kelvin wave analyzed?

Au: Yes.

page 14810, Eq. (19): if only the Kelvin wave was analyzed, Eq. (19) should be revised with $\left\|\underline{k}\right\| = \left(k_x^2 - a^2 k_y^2\right)^{1/2}$.

Au: Yes, thanks that was a typo.

page 14811, line 11-14 and Figure 4: the authors state: ``A drawback of the method is that it does not distinguish maxima due to the turbulent contribution and from the (presumed) wave contribution and in the empirical case, the separation is not always evident.'' The agreement between the theoretical dispersion curve (black) and empirically estimated one (blue) appears only at small wave numbers (i.e., for long waves). Does the black (or blue) line represent the Kelvin wave? If so, how can real k_y be possible for the Kelvin waves? As discussed in Eqs. (5) and (6) below, the black line in Figure 4 should be revised with the usage of $\|\underline{k}\| = \left(k_x^2 - a^2 k_y^2\right)^{1/2}$.

Au: Figure 4 shows the local maxima in the data (for ω fixed at different values) and compare it with singularities from eq.(19), associated with wave behaviour. In this framework, the form of the propagator is very general (with only a few constraints) and we used a simple form in eq.(19) (the black curve in fig.4) which leads to a dispersion relation similar to Kelvin waves. To provide an accurate description of Kelvin waves, we can also reproduce the fact they are "channeled" in the zonal direction with a slight modification of the wave propagator used in eq.(19) (see the discussion in the paragraph above eq.(18)).

page 14811, Figure 4: The Kevin wave (grey in Figure 3) appears between $\omega=(4)^{-1}$ and $\omega=(100)^{-1}$. However, selected values of ω are 2, 3, 5, and 10 h^{-1} in Figure 4. Please explain why these values are selected and then add discussions on the differences among different panels with different values of ω. In addition, what are the corresponding periods? 1/ω or 2π/ ω.

Au: Figure 4 is an attempt to detect the region of local maxima in the full 3D (k_x, k_y, ω) space, (associated with wave behaviour) and compare it with the singularity surface obtained from eq. (19). Ideally, this region should be visualized in 3D over the full ranges of (k_x, k_y, ω) values. Since it is more convenient to provide 2D graphs in a paper, different values of omega over its range were chosen in order to give the reader an idea of the shape of this maxima region in 3D.

In contrast, figure 3 shows the residual in the 2D (k_x, ω) space, after integration over k_y; so that the singularities were effectively integrated out (since $0<H_{wav}<1$). The grey region shows the local maxima, close to the Kelvin waves dispersion function. Indeed perfect coincidence would require the grey region to extend over the full range of omega.

The proper relation is $T=1/\omega$.

page 14813, line 7-10: the authors made the following strong statement: ``The main conclusion is thus that strongly turbulent atmospheric dynamics are a priori compatible with the observed waves, that to understand them, that one needn't invoke the existence of large laminar regimes nor linear theories." It is difficult for the reviewer to agree on the above statement because (1) only the Kelvin wave, which has a very special dispersion relation, was analyzed; (2) only limited data sets (for two months) were analyzed. Additional comments are given in the general comments (iii) and (iv).

Au: The referee is correct that the original paper essentially only analyzed data in the space-time region where Kelvin waves were likely to be found. With the addition of the new appendix C on the WKB approximation and gravity waves, the scope of the empirical treatment has been somewhat enlarged. However, the paper argues that these examples are typical of the general problem: linear wave theories are not likely to be compatible with the highly turbulent nature of the atmosphere. Indeed, we modestly stated:

"This paper is simply an early attempt to understand waves in highly turbulent media using scaling symmetries as constraints"

We have nevertheless modified the conclusions accordingly so that it now reads:

"Therefore, this paper should be seen more as a proof of concept than as providing definitive results. The main conclusion is thus that a priori, strongly turbulent atmospheric dynamics are compatible with the observed waves. If this is true, to understand them, requires neither the existence of large laminar regimes nor linear theories."