Zhuojian Yuan
Department of Atmospheric Science
Sun Yat-sen University
Guangzhou, Guangdong 510275
P. R. China
April 16, 2010

Dear Reviewer # 1:

Thank you very much for reminding us the improper vertical momentum equation (with the Earth's radius $r = a = 6.37 \times 10^6$m which leads to the absence of $\cos\phi$ Coriolis effect) used in the original manuscript (ms) entitled "*The basic mechanism behind the hurricane-free warm tropical ocean*" (acp-2009-742).

We decide to give up the improper equation and to adopt the completely-accurate equation with $r \neq a$. As pointed out by Phillips [please see the attached page 626 from Phillips' paper (1966) as well as page 401 from White and Bromley's paper (1995)], so long as $r \neq a$ is considered, the $2\Omega u \cos\phi$ should be included in the vertical momentum equation to ensure that this equation is completely accurate while building completely-accurate global models is the destination of atmospheric science. A large number of studies have emphasized the non-negligible effect of $\cos\phi$ Coriolis terms on the outputs of global models, the synoptic-scale systems in the tropics, the turbulent kinetic-energy budget in the oceanic surface mixed layer, Ekman layer stability, boundary layer eddies and nonhydrostatic mesoscale atmospheric systems etc. (e.g., Garwood et al., 1985; Leibovich and Lele, 1985; Draghici, 1987, 1989; Mason and Thomson, 1987; Shutts, 1989; Burger and Riphagen, 1990; White and Bromley, 1995). The condition for neglecting $\cos\phi$ Coriolis effect in the vertical momentum equation is $(2\Omega)^2 \ll N^2$ [please see the attached page 1156 from Phillips' paper (1968) as well as page 409 from White and Bromley's paper (1995)] where $\Omega = 7.292 \times 10^{-5}\,\text{s}^{-1}$ stands for the angular rotation rate of the Earth and

N for buoyancy frequency.

Our study focuses on destroying balance situation and creating perturbations in a completely-resting atmosphere for the embryo initiation (characterized by strong vertical convection with significant upward acceleration $dw/dt > 0$). The completely-resting atmosphere on the rotating Earth which we emphasize is characterized by zero relative winds ($u = v = w = 0$), zero buoyancy frequency ($N = 0$) and $(2\Omega)^2 > N^2 = 0$ which is opposite to $(2\Omega)^2 \ll N^2$. Therefore, $2\Omega u \cos 0^o = 2\Omega u$ at the Equator might not be neglected. To draw readers' attention immediately to this point, we replace the fourth paragraph of introduction in the original ms:

> *The above studies might lead to a more basic question of how, starting with an* ***internal-disturbance-free*** *balance-situation, external forces create the rapidly-upward acceleration of moist air at the warm sea surface. The reason for emphasizing the role of external forces in creating the disturbance is to avoid being dragged into the "chicken-and-egg" problem caused by the interactions of internal-forcing processes in the nonlinear atmospheric system. In the present study, two kinds of balance situations are considered. One is the idealized-balance situation characterized by the motionless field (e.g., without the presence of trade easterlies). The other is the wave-free trade easterlies between the Northern Hemisphere (NH) and the Southern Hemisphere (SH) subtropical-high belts under the geostrophic balance and hydrostatic balance. The wave-free trade easterlies refer to the zonally-homogeneous trade easterlies without easterly waves and the upstream-and-downstream effects of dispersive waves. The geostrophic balance and hydrostatic balance refer to the absence of horizontal and vertical accelerations and rising motion at the sea surface.*

with the following revised paragraph (all the revised sentences are in green):

> *The above studies might lead to a more basic question of how, starting with a completely-resting atmosphere on the rotating Earth ($\Omega = 7.292 \times 10^{-5}$ s^{-1})* ***with zero relative winds ($u = v = w = 0$), zero buoyancy frequency ($N = 0$)***

and $(2\Omega)^2 > N^2 = 0$, *external forces create the rapidly-upward acceleration of moist air at the warm sea surface. The reason for emphasizing the role of external forces in creating the disturbance for the embryo initiation is to avoid being dragged into the "chicken-and-egg" problem caused by the interactions of internal-forcing processes in the nonlinear atmospheric system.* ***The reason for emphasizing*** $(2\Omega)^2 > N^2 = 0$ ***is to make sure that the condition*** $(2\Omega)^2 \ll N^2$ ***required for neglecting the*** $\cos\phi$ ***Coriolis effect is not satified around the Equator (e.g., Phillips, 1968; White and Bromley, 1995).*** *In the present study, two kinds of balance situations are considered. Besides the idealized-balance situation characterized by* $u = v = w = 0$ *and* $N = 0$, *the second one is a more realistic balance situation characterized by the wave-free trade easterlies between the Northern Hemisphere (NH) and the Southern Hemisphere (SH) subtropical-high belts under the geostrophic balance and hydrostatic balance with* $N = 0$. *The wave-free trade easterlies refer to the zonally-homogeneous trade easterlies without easterly waves and the upstream-and-downstream effects of dispersive waves. The geostrophic balance and hydrostatic balance refer to the absence of horizontal acceleration and vertical acceleration, the absence of buoyancy and vertical pressure gradient perturbation as well as the absence of rising motion at the sea surface.*

****

Please allow us to say a few more words about the above original paragraph and revised paragraph.

In the above original paragraph, the expression "***internal-disturbance-free***" also includes the zero effect of the vertical pressure gradient perturbation. The zero effect of the vertical pressure gradient perturbation also means zero buoyancy frequency since the hydrostatic balance is:

$$-\frac{1}{\rho}\frac{\partial p}{\partial z}-g=-\frac{1}{\rho}\frac{\partial\left(\overline{p}+p'\right)}{\partial z}-g$$
$$=-\frac{1}{\rho}\frac{\partial\overline{p}}{\partial z}-g-\frac{1}{\rho}\frac{\partial p'}{\partial z}=0.$$

With the use of $\partial\overline{p}/\partial z=-\overline{\rho}g$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z}-g$$
$$=\frac{1}{\rho}\overline{\rho}g-g-\frac{1}{\rho}\frac{\partial p'}{\partial z}$$
$$=-g\frac{\rho-\overline{\rho}}{\rho}-\frac{1}{\rho}\frac{\partial p'}{\partial z}$$
$$=-N^2\delta z-\frac{1}{\rho}\frac{\partial p'}{\partial z}=0.$$

Therefore if $N=0$ then the vertical pressure gradient perturbation is also zero.

To avoid burying all the above information in the original expression "***internal-disturbance-free***", we symbolize the original expression "***internal-disturbance-free***" by "**zero relative winds ($u=v=w=0$), zero buoyancy frequency ($N=0$) and** $(2\Omega)^2>N^2=0$" in the above revised paragraph. We hope this symbolization will make our point easy to follow that $\cos\phi$ Coriolis effect around the Equator is not negligible since $(2\Omega)^2>N^2=0$ is opposite to the condition $(2\Omega)^2\ll N^2$ required for neglecting the $\cos\phi$ Coriolis effect (e.g., Phillips, 1968; White and Bromley, 1995).

****

To ensure that the horizontal momentum equations and the mass continuity equation are spelt out explicitly in the paragraph about how hurricane's embryo is generated from the second (more realistic) kind of balance situation (the first kind is idealized one with $u=v=w=0$ and $N=0$), we revise the paragraph as following:

Now we turn to the second kind of balance situation with the presence of trade easterlies which is more realistic than the first one. In reality, the embryo initiation *is a complicated nonlinear process due to the nonlinear term* $w\partial w/\partial r$ *in the definition of* dw/dt. *Additional nonlinear processes indicated by nonlinear terms such as* $u\partial u/\partial x$ ***in the horizontal momentum equations*** *will come into play due to the presence of trade easterlies in the real atmosphere. In such a more realistic case, the contribution to the vertical motion (not to the vertical acceleration in a direct way) could also be attributed to the physical processes described by* ***the horizontal momentum equations*** *through the linkage between the low-level equatorial convergence* $\nabla_2 \cdot \boldsymbol{V} < 0$ *and the rising motion determined by* ***the mass-continuity equation***. *Although the additional nonlinear processes such as* $u\partial u/\partial x$ *might make the embryo-initiation process more complicated, they could enlarge the effect of LLEWW embedded in trade easterlies on* $w > 0$ *at the sea surface due to the additional effect of low-level equatorial convergence zone (Anthes, 1982, 49-51). This well-known low-level equatorial convergence zone would be able to create the upward transport of moist air from the warm sea surface to the midtroposphere under the condition that the negative effect of other internal disturbances is small as compared with the effect of the equatorial convergence zone. This low-level equatorial convergence zone together with* $dw/dt = 2\Omega u > 0$ *at the Equator might account for most convective clouds observed around the Equator under the effects of the largest continent with the highest plateau and the largest ocean with the warm pool located to the east and on the equatorward side of the continent on the rotating Earth (Fig. 2).*

We summarize all the above changes in the revised section 4. The revised sentences are organized as following:

4. The linkage between the LLEWW and the upward acceleration

Focusing on the mechanisms for destroying the balance situation characterized by a completely-resting relative-wind field with zero vertical velocity ($w = 0$ *), zero horizontal velocities (*$u = v = 0$*) and zero buoyancy frequency (* $N = 0$*), we turn to* *the external processes for creating upward acceleration (* $dw/dt > 0$ *) of vapour at the sea surface. According to the completely-accurate motion equations (derived in the Earth's spherical coordinates) considered by Phillips (1966) as well as by other scientists (e.g., White and Bromley, 1995; White, 2002, p. 14), the creation of vertical acceleration at the sea surface is attributed to five processes:*

$$\frac{dw}{dt} = \underset{1}{2\Omega u\cos\phi} - \underset{2}{\frac{1}{\rho}\frac{\partial p}{\partial r}} - \underset{3}{g} + \underset{4}{\frac{u^2+v^2}{r}} + \underset{5}{F_{rz}} , \tag{1}$$

where the Earth's radius r *is not the constant* $a = 6.37\times10^6$ *m. The five processes include the combined effect of the* $\cos\phi$ *Coriolis parameter and the relative zonal velocity (term 1). So long as external forces generate equatorial (* $\cos 0^o = 1$ *) zonal wind,* $2\Omega u$ *could be as large as the midlatitude* $\sin\phi$ *Coriolis term i.e.,* $2\Omega u\sin\left(45^o\right) = \sqrt{2}\Omega u$ *receiving most attention due to its significant role in the horizontal momentum equation. A large number of studies have emphasized the non-negligible effect of* $\cos\phi$ *Coriolis terms on the outputs of global models, the synoptic-scale systems in the tropics, the turbulent kinetic-energy budget in the oceanic surface mixed layer, Ekman layer stability, boundary layer eddies and nonhydrostatic mesoscale atmospheric systems etc. (e.g., Garwood et al., 1985; Leibovich and Lele, 1985; Draghici, 1987, 1989; Mason and Thomson, 1987; Shutts, 1989; Burger and Riphagen, 1990; White and Bromley, 1995). Besides this* $\cos\phi$ *Coriolis term in Eq. (1), the contribution to* $dw/dt \neq 0$ *is also attributed to the vertical pressure gradient force (term 2), the gravity (term 3), the Earth's curvature effect (term 4) and the frictional force (term 5).*

The completely-resting background with $N = 0$ *emphasized in the present embryo-initiation study is at least in the hydrostatic balance:*

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} - g = 0 . \tag{2}$$

As a result, the contribution to $dw/dt \neq 0$ *might come from the following three* *right-hand side (rhs) terms:*

$$\frac{dw}{dt} = 2\Omega u \cos\phi + \frac{u^2 + v^2}{r} + F_{rz} \tag{3}$$

for destroying the balance situation and making the hurricane's embryo originate in the completely-resting atmosphere on the rotating Earth due to external forces. According to Holton (2004, p. 41), the magnitudes of the last two rhs terms of Eq. (3) are much smaller than the magnitude of the first rhs term. So the last two rhs terms can be eliminated, leading to:

$$\frac{dw}{dt} = 2\Omega u \cos\phi. \tag{4}$$

In Eq. (4), Ω is the angular rotation rate of the Earth while the Earth's rotation is the well-known source of external force acting on the atmosphere. Be aware that if Ω were zero or horizontal velocity were absent (White 2002, p. 23) together with zero horizontal convergence i.e., $\nabla_2 \cdot \boldsymbol{V} = 0$, then hydrostatic balance would lead to the zero vertical acceleration:

$$\frac{dw}{dt} = 0. \tag{5}$$

Without the vertical acceleration and without the horizontal convergence, the moist air at the sea surface with $w = 0$ could not be transported to the middle troposphere. However in reality, the angular rotation rate of the Earth $\Omega = 7.292 \times 10^{-5}$ s^{-1} is non-zero, so is the external-force-induced LLEWW ($u > 0$) due to processes such as the deflection of the cross-equatorial wind induced by the land-sea differential heating. Therefore Eq. (4) reveals that even beginning with a completely-resting atmosphere on the rotating Earth with zero relative wind field ($u = v = w = 0$) and $(2\Omega)^2 > N^2 = 0$, the external-force-induced LLEWW-burst ($u > 0$) can boost upward acceleration in the latitudes with $\cos\phi \neq 0$.

Again we draw attention to the following four facts. Firstly, $\cos\phi$ reaches its maximum value (i.e., $\cos 0^o = 1$) at the Equator and the relative velocities u and w detected on the rotating Earth are in the same equatorial plane with $\boldsymbol{\Omega}$

perpendicular to the vertical coordinate $\boldsymbol{k}_{\text{eq}} = \boldsymbol{w}_{\text{eq}} / |\boldsymbol{w}_{\text{eq}}|$ *(Fig. 3). Secondly,* $dw/dt = 2\Omega u > 0$ *at the Equator created by the external-force-induced evenly-distributed flow (double-line arrows in Fig. 3) could be at least as significant as the midlatitude Coriolis force* $\sqrt{2}\Omega u$ *which receives tremendous attention. Thirdly, the external-force-induced LLEWW does exist in the real atmosphere. We focus on such a LLEWW to avoid the "chicken-and-egg" problem accompanying the nonlinear interactions of internal processes. Avoiding the "chicken-and-egg" problem might lessen the difficulty of extracting the leading signal carried by the external-force-induced LLEWW according to Eq. (4). Fourthly, although the magnitude of external-force-induced* $2\Omega u$ *is smaller than that of the gravity, the gravity is always balanced more or less by the vertical pressure gradient force. So long as the horizontal wind i.e., LLEWW (* $u > 0$ *) comes into play, the contribution of* $2\Omega u$ *(despite being abandoned as a residual term) to* $dw/dt > 0$ *(commonly observed in the convective systems around the Equator) might be greater than the net contribution of the gravity and vertical pressure gradient forces especially for destroying the completely-resting atmosphere on the rotating Earth with* $(2\Omega)^2 > N^2 = 0$. *According to Phillips (1968),* $(2\Omega)^2 \ll N^2$ *is required for neglecting the* $\cos\phi$ *Coriolis term. Thus in our opinion, neglecting the external-force-induced* $2\Omega u \cos\phi$ *around the Equator might lead to unrealistic results of the study focusing on the origination of hurricane's embryo from a completely-resting atmosphere on the rotating Earth with* $(2\Omega)^2 > N^2 = 0$.

As mentioned in the introduction, two kinds of balance situations are considered in the present study. The first one is the idealized-balance situation characterized by a completely-resting atmosphere on the rotating Earth with zero relative winds ($u = v = w = 0$ *e.g., without the presence of trade easterlies) and zero buoyancy frequency (* $N = 0$ *). The second one is a more realistic balance situation characterized by the wave-free trade easterlies between the NH and SH subtropical-high belts under the geostrophic balance and hydrostatic balance with*

$N = 0$. *In the first kind of balance situation,...*

Now we turn to the second kind of balance situation with the presence of trade easterlies which is more realistic than the first one. In reality, the embryo initiation is a complicated nonlinear process due to the nonlinear term $w\partial w / \partial r$ *in the definition of* dw / dt. *Additional nonlinear processes indicated by nonlinear terms such as* $u\partial u / \partial x$ *in the horizontal momentum equations will come into play due to the presence of trade easterlies in the real atmosphere. In such a more realistic case, the contribution to the vertical motion (not to the vertical acceleration in a direct way) could also be attributed to the physical processes described by the horizontal momentum equations through the linkage between the low-level equatorial convergence* $\nabla_2 \cdot \boldsymbol{V} < 0$ *and the rising motion determined by the mass-continuity equation. Although the additional nonlinear processes such as* $u\partial u / \partial x$ *might make the embryo-initiation process more complicated, they could enlarge the effect of LLEWW embedded in trade easterlies on* $w > 0$ *at the sea surface due to the additional effect of low-level equatorial convergence zone (Anthes, 1982, 49-51). This well-known low-level equatorial convergence zone would be able to create the upward transport of moist air from the warm sea surface to the midtroposphere under the condition that the negative effect of other internal disturbances is small as compared with the effect of the equatorial convergence zone. This low-level equatorial convergence zone together with* $dw / dt = 2\Omega u > 0$ *at the Equator might account for most convective clouds observed around the Equator under the effects of the largest continent with the highest plateau and the largest ocean with the warm pool located to the east and on the equatorward side of the continent on the rotating Earth...*

Sincerely,

Zhuojian Yuan

Professor

NOTES AND CORRESPONDENCE

The Equations of Motion for a Shallow Rotating Atmosphere and the "Traditional Approximation"

NORMAN A. PHILLIPS[1]
National Center for Atmospheric Research, Boulder, Colo.

20 April 1966

A long-standing point of confusion in atmospheric dynamics has been the role of the coriolis terms proportional to the cosine of the latitude. In hydrostatic problems they are ignored for very logical reasons, but, as pointed out by Eckart (1960), their significance is not so clear in non-hydrostatic problems. This note provides a rationale for discarding them in quite general motions of a *shallow* atmosphere.

Let r (radius), λ (longitude) and φ (latitude) represent spherical coordinates. Because of the small ellipticity of the earth, these coordinates, with "sea-level" located at $r=a=$ constant, and "gravity" having no φ-component, are reasonably exact for analyzing atmospheric motions. The relative velocity components are

$$\left.\begin{aligned} u&=h_\lambda\frac{d\lambda}{dt}=r\cos\varphi\frac{d\lambda}{dt}\\ v&=h_\varphi\frac{d\varphi}{dt}=r\frac{d\varphi}{dt}\\ w&=h_r\frac{dr}{dt}=1\frac{dr}{dt}\end{aligned}\right\}, \qquad (1)$$

while the equations of motion are usually written as

$$\left.\begin{aligned} \frac{du}{dt}&=F_\lambda+\left(2\Omega+\frac{u}{r\cos\varphi}\right)(v\sin\varphi-w\cos\varphi)\\ \frac{dv}{dt}&=F_\varphi-\left(2\Omega+\frac{u}{r\cos\varphi}\right)u\sin\varphi-\frac{wv}{r}\\ \frac{dw}{dt}&=F_r-g+\left(2\Omega+\frac{u}{r\cos\varphi}\right)u\cos\varphi+\frac{v^2}{r}\end{aligned}\right\}. \qquad (2)$$

F_λ, F_φ and F_r are the three components of the pressure and frictional forces per unit mass, while g represents both the Newtonian gravitational attraction of the earth and the centrifugal acceleration due to Ω. These equations are not exact because they include the assumption that the ellipticity of the earth is small, so that near the surface of the earth geographic latitude differs only slightly from geocentric latitude. This note is not concerned with this approximation, however, but with a further simplification usually introduced into (2) for mathematical convenience. Nonetheless, it is important to note here that the approximations leading to (2) have not corrupted the angular momentum principle; (2) and (1) together imply that

$$\frac{d}{dt}[r\cos\varphi(u+\Omega r\cos\varphi)]=r\cos\varphi F_\lambda. \qquad (3)$$

From this point on, (1) and (2) are considered as completely accurate.

The *shallowness* of the atmosphere (as far as meteorologists are concerned) certainly seems to justify the practice of replacing r as a coefficient by a (and $\partial/\partial r$ by $\partial/\partial z$) in the hydrodynamic equations (2). It appears that this has usually been done, either explicitly or implicitly, by simply replacing (1) and (2), respectively, by

$$\left.\begin{aligned} u'&=a\cos\varphi d\lambda/dt\\ v'&=a\, d\varphi/dt\\ w'&=dz/dt\end{aligned}\right\}, \qquad (1')$$

and

$$\left.\begin{aligned} \frac{du'}{dt}&=F_\lambda+\left(2\Omega+\frac{u'}{a\cos\varphi}\right)(v'\sin\varphi-w'\cos\varphi)\\ \frac{dv'}{dt}&=F_\varphi-\left(2\Omega+\frac{u'}{a\cos\varphi}\right)u'\sin\varphi-\frac{w'v'}{a}\\ \frac{dw'}{dt}&=F_z-g+\left(2\Omega+\frac{u'}{a\cos\varphi}\right)u'\cos\varphi+\frac{v'^2}{a}\end{aligned}\right\}. \qquad (2')$$

[See, for example, Haltiner and Martin (1957, pp. 166–168) and Eckart (1960, p. 100).]

These equations have the serious deficiency that they do not possess an angular momentum principle, i.e., there appears to be no function $A(\varphi,z)$ (other than zero) such that $AF_\lambda=dM/dt$ for arbitrary Ω, u', v', w', F_φ and F_z. The difficulty is caused by the w' terms on

[1] Permanent address: Department of Meteorology, Massachusetts Institute of Technology, Cambridge, Mass.

The attached page 626 from Phillips' paper (1966).

were used, the $\cos\varphi$ term would be absent. The formula is to be interpreted only as a WKBJ approximation to the complete partial differential equation, since the latter has variable coefficients. When $N^2 \gg 4\Omega^2$, it is difficult to have $\Omega \cos\varphi$ play a dominant role (at least when the wavenumbers α, β, γ are restricted to real values) since this requires large β^2, and then N^2 dominates. On the other hand, $\Omega \sin\varphi$ can be made dominant by simply choosing (α/γ) and (β/γ) to be sufficiently small, whereupon the so-called "inertia waves" result, in which $\omega \sim 2\Omega \sin\varphi$. [See also Bretherton (1964).]

Although this argument shows that it is difficult to ascribe any *dominant* importance to the $2\Omega \cos\varphi$ term when $N^2 \gg 4\Omega^2$ this term can contribute in more subtle ways. For example, Backus (1962) has shown that although the theoretical effect of the Coriolis force on the propagation of oceanic surface waves is very small, the small Coriolis effect enters primarily through the $2\Omega \times \cos\varphi$ term rather than through $2\Omega \sin\varphi$. A second example in which $2\Omega \cos\varphi$ may enter significantly *in the finer details* occurs in the "inertia waves" referred to above. To see this, it is convenient to rewrite (6) as

$$\omega^2 = \frac{4\Omega^2 \sin^2\varphi + N^2\left(\dfrac{\alpha^2+\beta^2}{\gamma^2}\right) + 8\Omega^2 \sin\varphi \cos\varphi\left(\dfrac{\beta}{\gamma}\right) + 4\Omega^2 \cos^2\varphi\left(\dfrac{\beta}{\gamma}\right)^2}{1+(\alpha^2+\beta^2)/\gamma^2}. \tag{7}$$

The third and fourth terms in the numerator are the $2\Omega \cos\varphi$ terms. Smallness of α/γ and β/γ gives $\omega \sim 2\Omega \times \sin\varphi$, as mentioned earlier. But the *deviation* of ω from $2\Omega \sin\varphi$ due to a nonzero value of (β/γ) can be influenced by the third term in the numerator as much as by the $N^2\beta^2/\gamma^2$ term if (β/γ) is small enough. A more careful examination of this point requires a solution of the complete partial differential equation, since (6) is a large wavenumber approximation. As noted by Eckart (1960, p. 101 and pp. 130–135), this solution is not a trivial matter because that partial differential equation is nonseparable in z and φ. [Some further remarks, unfortunately also inconclusive, can be found in an unpublished thesis by B. Hughes (1964) from Cambridge University, and in a paper by W. Munk and myself scheduled for the November 1968 issue of *Reviews of Geophysics*.]

A similar restraining influence of small $4\Omega^2/N^2$ on the importance of $2\Omega \cos\varphi$ *vis-à-vis* $2\Omega \sin\varphi$ occurs in the case of Rossby waves. To show this, it is necessary, however, to derive the Rossby frequency relation in a different manner than is to be found in the literature, since the usual derivations ignore the effect in question. A convenient starting point is the homogeneous form of the linear equations given by Eckart (1960, p. 56). The WKBJ formalism, in which variables are represented in the form $u = \mathrm{Re}[U \exp(i\psi)]$ and in which the common phase ψ is normally taken to vary rapidly in both space and time, is applied to these equations, but with special treatment of the frequency $\omega = -\partial\psi/\partial t$ compared to the wavenumber $\mathbf{k} = (\alpha, \beta, \gamma) = \nabla\psi$. For example, $\partial u/\partial t = e^{i\psi}(\partial U/\partial t - i\omega U)$ and $\nabla u = e^{i\psi}(\nabla U + i\mathbf{k}U)$ are expanded as follows, with subscripts and brackets indicating the successive approximations:

$$\frac{\partial U}{\partial t} - i\omega U = [\text{zero}] - [i\omega_1 U_0] + \left[\frac{\partial U_0}{\partial t} - i\omega_2 U_0 - i\omega_1 U_1\right] + \cdots$$

$$\nabla U + i\mathbf{k}U = [i\mathbf{k}_0 U_0] + [i\mathbf{k}_0 U_1 + i\mathbf{k}_1 U_0 + \nabla U_0] + \cdots.$$

In other words, ω_0 is set equal to zero, but the dominant time dependence is still due to ω. The result is most readily obtained by considering the zero- and first-order expansions of the five equations given by Eckart; thus,

$$\omega_1 = \frac{-\left(\dfrac{2\Omega \cos\varphi}{r}\right)\alpha_0}{\alpha_0^2+\beta_0^2+\dfrac{4\Omega^2}{N^2}\left[\left(\dfrac{N^2}{c_0^2}+\Gamma^2\right)\sin^2\varphi + (\gamma_0 \sin\varphi + \beta_0 \cos\varphi)^2\right]}, \tag{8}$$

where c_0 is the speed of sound in the undisturbed state and Γ is Eckart's compressibility parameter, $(g/c_0^2) + \frac{1}{2}d \times [\ln(\rho_0 c_0)]/dr$, in the undisturbed state ($\Gamma^2 + N^2/c_0^2$, however, is normally small compared to any reasonable value of γ_0^2). The Coriolis term $2\Omega \sin\varphi$ in the horizontal equations of motion is responsible for the $\sin\varphi$ terms in the denominator and, through its variation with latitude, for the numerator. The Coriolis term $2\Omega \cos\varphi$ introduces only the $\beta_0 \cos\varphi$ term in the denominator. Smallness of $4\Omega^2/N^2$ again prevents this term from contributing in any significant manner.

Veronis shows that the equations (2) possess an an-

The attached page 1156 from Phillips' paper (1968).

conservation properties are discussed in detail by Lorenz (1967), Phillips (1973) and Hoskins *et al.* (1985). Aspects which are important in this study are reviewed here.

(*a*) *The Navier–Stokes equation*

If velocities **u** are measured relative to a system rotating with angular velocity $\mathbf{\Omega}$, the Navier–Stokes equation is

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + 2\mathbf{\Omega} \times \mathbf{u} - \mathbf{g} + \frac{1}{\rho}\operatorname{grad} p = \mathbf{F}. \tag{2.1}$$

Here **g** is apparent gravity (true gravity plus the centrifugal term $-\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$, where **r** is position relative to the centre of the earth): $\mathbf{g} = -\operatorname{grad} \Phi$, where Φ is the geopotential, ρ is density, p is pressure and **F** is the frictional force per unit mass. The spheroidal geopotential surfaces are customarily represented as spheres. In spherical polar coordinates the three components of (2.1) are then

$$\frac{\mathrm{D}u}{\mathrm{D}t} - \left(2\Omega + \frac{u}{r\cos\phi}\right)(v\sin\phi - w\cos\phi) + \frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda} = F_\lambda \tag{2.2}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + \left(2\Omega + \frac{u}{r\cos\phi}\right)u\sin\phi + \frac{vw}{r} + \frac{1}{\rho r}\frac{\partial p}{\partial\phi} = F_\phi \tag{2.3}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} - \left(2\Omega + \frac{u}{r\cos\phi}\right)u\cos\phi - \frac{v^2}{r} + g + \frac{1}{\rho}\frac{\partial p}{\partial r} = F_r. \tag{2.4}$$

Here

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial r} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\operatorname{grad} \tag{2.5}$$

and u, v, w are the components of **u** in the λ (longitude), ϕ (latitude) and r (radial) directions; the polar axis is in the direction of $\mathbf{\Omega}$. The coordinate configuration is shown in Fig. 1.

When taken together with the continuity and thermodynamic equations

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\operatorname{div}\mathbf{u} = 0 \tag{2.6}$$

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \left(\frac{\theta}{Tc_p}\right)Q \tag{2.7}$$

Eqs. (2.2)–(2.5) imply the following conservation laws for axial angular momentum, energy and potential vorticity:

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\Big((u + \Omega r\cos\phi)r\cos\phi\Big) = \rho F_\lambda r\cos\phi - \frac{\partial p}{\partial\lambda} \tag{2.8}$$

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\mathbf{u}^2 + \Phi + c_vT\right) + \operatorname{div}(p\mathbf{u}) = \rho(Q + \mathbf{u}\cdot\mathbf{F}) \tag{2.9}$$

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\mathbf{Z}\cdot\operatorname{grad}\theta}{\rho}\right) = \mathbf{Z}\cdot\operatorname{grad}\frac{\mathrm{D}\theta}{\mathrm{D}t} + \operatorname{grad}\theta\cdot\operatorname{curl}\mathbf{F}. \tag{2.10}$$

Perfect gas behaviour, $p = \rho RT$, has been assumed. In Eq. (2.10), **Z** is the absolute vorticity, $2\mathbf{\Omega} + \operatorname{curl}\mathbf{u}$. The components of **Z** in the (λ, ϕ, r) system are

The attached page 401 from White and Bromley's paper (1995).

$$\left|\frac{\mathrm{D}w}{\mathrm{D}t}\right| \sim \frac{UW}{L} \leqslant \frac{U^2H}{L^2}$$

so

$$\frac{|\mathrm{D}w/\mathrm{D}t|}{|2\Omega u|} \leqslant \frac{UH}{2\Omega L^2} = \frac{H}{L} Ro \sim 10^{-3}$$

assuming $L = 10^6$ m, $U = 10\,\mathrm{m\,s^{-1}}$ and $H = 10^4$ m. Draghici (1987, 1989) notes that $-2\Omega u \cos\phi$ dominates $\mathrm{D}w/\mathrm{D}t$ for a range of mesoscale motions also, and thus apparently represents the most important nonhydrostatic effect in such cases.

(c) *Previous adiabatic analyses*

The above scale analysis suggests that the treatment of dynamically important balances in Eqs. (2.2) and (2.4) may be subject to errors of about 10% (at least in the tropics, or on planetary scales) if the $\cos\phi$ Coriolis terms are neglected. Phillips (1968) and Gill (1982) have considered these terms to be less important. From an approximate dispersion relation for linearized waves in an atmosphere at rest, Phillips identified $4\Omega^2 \ll N^2$ (where N is the buoyancy frequency) as the condition for neglect of the $\cos\phi$ Coriolis terms. Gill gave the more stringent condition $2\,\Omega \ll N$ after a scale analysis of the linearized, equatorial β-plane equations. Either condition reveals an unsatisfactory aspect of the HPEs: they do not remain a physically acceptable approximation as the static stability (and hence N) tends to zero. Nevertheless, if N is of order $10^{-2}\mathrm{s}^{-1}$ (a typical tropospheric value) then $2\Omega/N \sim 10^{-2}$, and so the $\cos\phi$ Coriolis terms are negligible even according to Gill's criterion.

Both Phillips's and Gill's analyses assume adiabatic motion, vertical velocities being related to horizontal density fluctuations and the buoyancy frequency. In deriving the condition $2\Omega H \cos\phi/U \ll 1$ for the neglect of the $\cos\phi$ Coriolis term in Eq. (2.2) (see Eq. (3.2)), we have estimated vertical velocities from the continuity equation and have thus used an upper bound which may be approached in regions of strong diabatic heating. This seems a suitable treatment for tropical synoptic-scale convective complexes in which diabatic heating plays an important role in the dynamics (including the determination of phase speeds). Neither Phillips's nor Gill's analysis is applicable to planetary-scale motion.

From the above discussion we conclude that the $\cos\phi$ Coriolis terms cannot comfortably be neglected. Amongst motions for which they may play a small but not negligible role are synoptic-scale, diabatically driven flows in the tropics, and planetary-scale flows. We therefore proceed to examine extensions of the HPEs in which the $\cos\phi$ Coriolis terms are retained.

4. Extensions of the hydrostatic primitive equations that conserve energy, angular momentum and potential vorticity

Some authoritative texts (e.g. Holton 1972; Gill 1982) cite shallow-atmosphere forms of the components of the Navier–Stokes equations in which the $\cos\phi$ Coriolis terms and all of the metric terms are retained. These forms conserve energy but do not imply precise analogues of angular-momentum and potential-vorticity conservation. Indeed, it is clear that inclusion of the $\cos\phi$ Coriolis terms can never be fully consistent with the shallow-atmosphere approximation: as discussed in section 3(a), the term $2\Omega w \cos\phi$ in Eq. (2.2) represents the height variation of planetary angular momentum, which is suppressed in the shallow-atmosphere approximation. Since we consider it desirable that

The attached page 409 from White and Bromley's paper (1995).

References:

Anthes, R. A.: Tropical cyclones: Their evolution, structure and effects, American Meteorological Society, Ephrata, PA, 208 pp., 1982.

Burger, A. P., and Riphagen, H. A.: The basic equations in meteorological dynamics - A reexamination of unsimplified forms for a general vertical coordinate, Beitr. Phys. Atmos., 63, 151-164, 1990.

Draghici, I.: Non-hydrostatic Coriolis effects in an isentropic coordinate frame, Meteorol. Hydrol., 17, 45-54, 1987.

Draghici, I.: The hypothesis of a marginally-shallow atmosphere, Meteorol. Hydrol., 19, 13-27, 1989.

Garwood, R. W., Gallacher, P. C., and Muller, P.: Wind direction and equilibrium mixed layer depth: General theory, J. Phys. Oceanogr., 15, 1325-1331, 1985.

Holton, J. R.: An introduction to dynamic meteorology, 4th ed., International geophysics series, Elsevier Academic Press, New York, 535 pp., 2004.

Leibovich, S., and Lele, S. K.: The influence of the horizontal component of Earth's angular velocity on the instability of the Ekman layer, J. Fluid Mech., 150, 41-87, 1985.

Mason, P. J., and Thomson, D. J.: Large-eddy simulations of the neutral-static-stability planetary boundary layer, Q. J. R. Meteorol. Soc., 113, 413-443, 1987.

Phillips, N. A.: The equations of motions for a shallow rotating atmosphere and the "tranditional approximation", J. Atmos. Sci., 23, 626-628, 1966.

Phillips, N. A.: Reply to G. Veronis's comments on Phillips (1966), J. Atmos. Sci., 25, 1155-1157, 1968.

Shutts, G. J.: Planetary semi-geostrophic equations derived from Hamilton's principle, J. Fluid Mech., 208, 545-573, 1989.

White, A. A., and Bromley, R. A.: Dynamically consistent, quasi-hydrostatic equations for global models with a complete representation of the Coriolis force, Q. J. R. Meteorol. Soc., 121, 399-418, 1995.

White, A. A.: A view of the equations of meteorological dynamics and various approximations, in: Large-scale atmosphere-ocean dynamics I: Analytical methods and numerical models, edited by: Norbury, J., and Roulstone, I., Cambridge University Press, Cambridge, 1-100, 2002.